Introduction to
Linear Circuit Analysis and Modelling

From DC to RF

Solutions Manual

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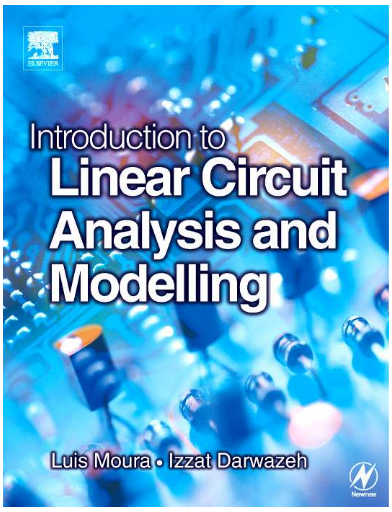

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Chapter 1

Elementary electrical circuit analysis

Solution of problem 1.1

The current flowing through the capacitor is given by:

$$\begin{aligned} i(t) &= C\,\frac{d\,v(t)}{dt} \\ &= 10^{-6}\,2\,\pi\,1000\cos(2\,\pi\,100\,t+\pi/4)\ \text{ A} \\ &= 6.3\ \cos(2\,\pi\,100\,t+\pi/4)\ \text{ mA} \end{aligned}$$

Figure 1.1 shows the waveforms for the voltage across and the current through the capacitor.

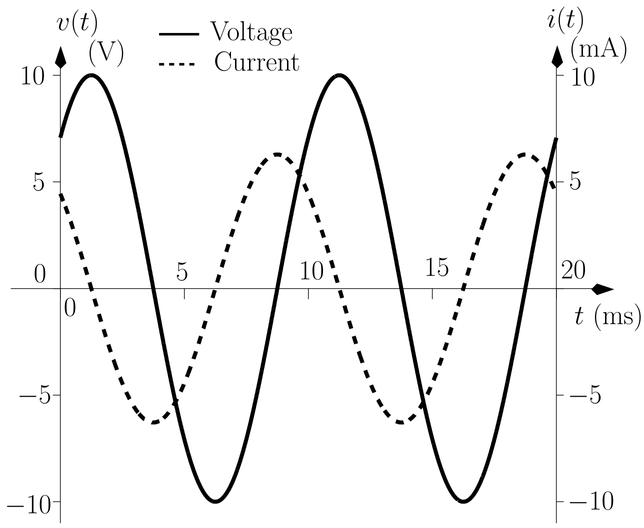


Figure 1.1: *Waveforms for the voltage across and the current through the capacitor.*

Solution of problem 1.2

The voltage across the terminals of the inductor is given by:

$$\begin{aligned} v(t) &= L\,\frac{d\,i(t)}{dt} \\ &= -3\times 10^{-3}\,2\,\pi\,5000 \times 20\times 10^{-3}\sin(2\,\pi\,5000\,t)\ \ \mathrm{V} \\ &= -1.9\,\sin(2\,\pi\,5000\,t)\ \ \mathrm{V} \end{aligned}$$

Figure 1.2 shows the waveforms for the voltage across and the current through the inductor.

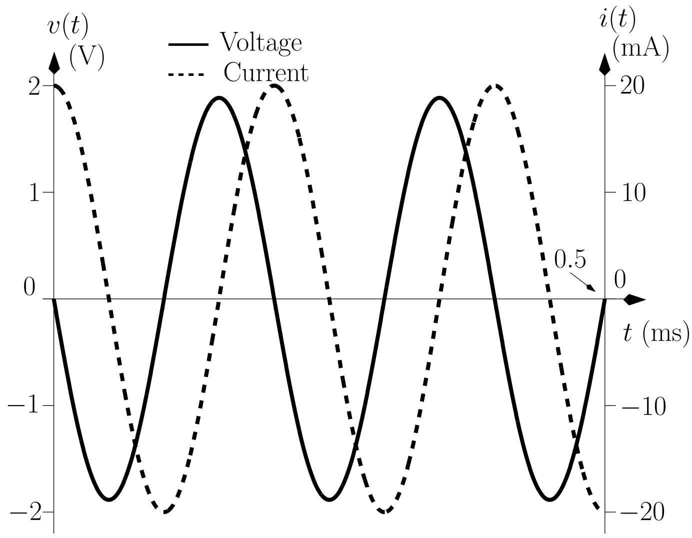


Figure 1.2: *Waveforms for the voltage across and the current through the inductor.*

Solution of problem 1.3

The calculations of the voltages across and the currents through each circuit elements are effected applying the Nodal analysis. This method can be outlined as follows:

1. First, we indicate the voltages at each node. These voltages indicate the potential difference between the node being considered and a reference node which can be chosen arbitrarily. This node is traditionally called 'node zero' (0) or the 'ground terminal' and is often chosen as the node with the highest number of attached electrical elements. This is illustrated in figure 1.3 a)

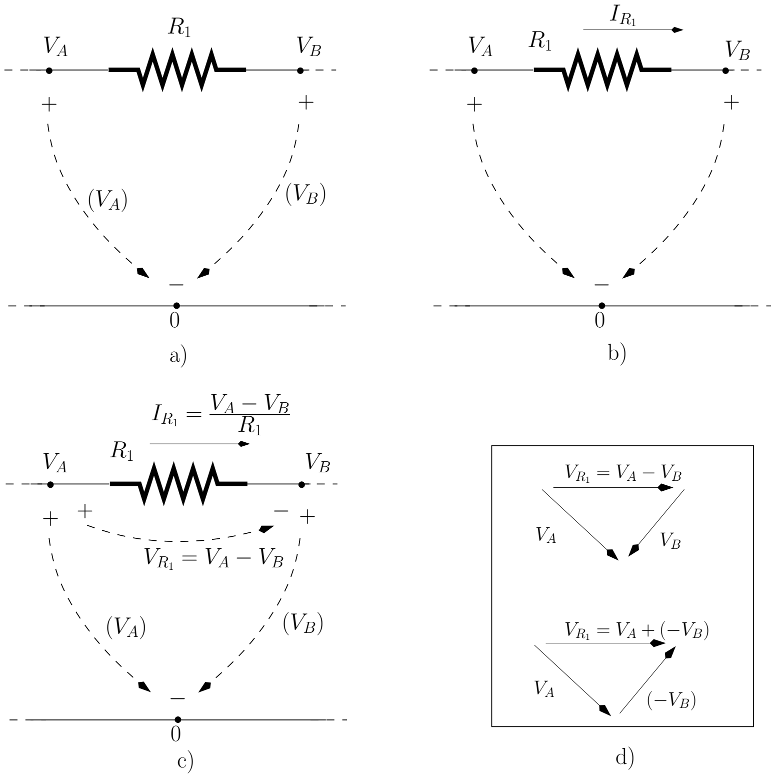


Figure 1.3: *Application of the Nodal analysis method. a) Indication of the voltage is the nodes A and B b) Indication of the current I_{R_1} c) $I_{R_1} = (V_A - V_B)/R$. d) Adding vectors.*

2. Then, we consider, in an arbitrary manner, the current direction in each branch, as indicated in figure 1.3 b).

3. The current that flows through each resistance can be expressed, according to Ohm's law, as the ratio of the voltage across that resistance and the resistance value;

$$I_{R_1} = \frac{V_A - V_B}{R_1} \tag{1.1}$$

   Note that the similarity between the way we express the voltage across the resistance, as $V_A - V_B$, and the calculation of the sum (or subtraction) of vectors shown in figure 1.3 d).

4. Finally, we apply the current voltage law to each node[1]

We apply now the Nodal analysis method to solve the circuits of the problem 1.3 where the voltages at each node and the directions of the currents have been chosen as indicated in figure 1.4.

[1]As its name suggests the Nodal analysis method is based on the application of the current voltage law to each node of the circuit. However, the calculation of some circuits may require the application of the voltage law.

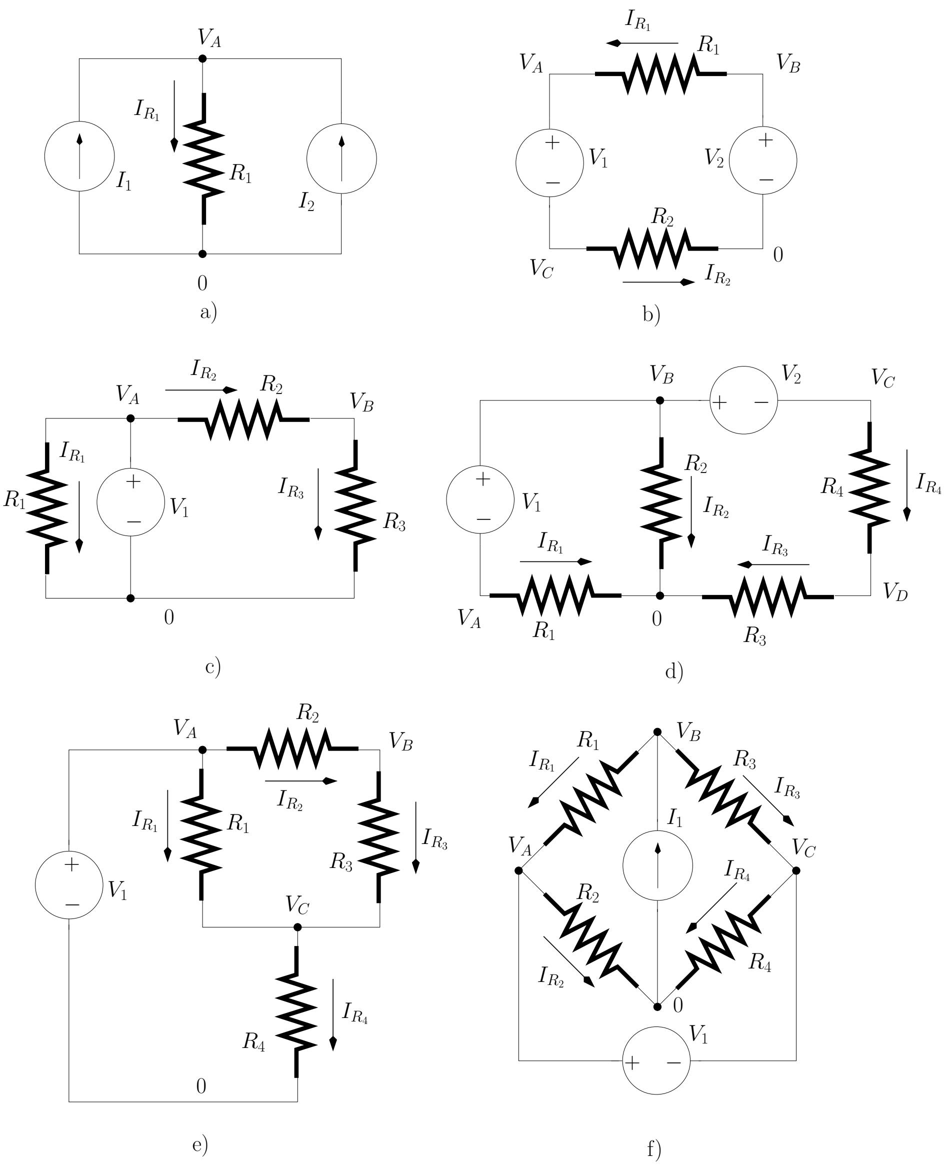


Figure 1.4: *Circuits of problem 1.3.*

- *Circuit a):* Applying the current law we can write:

$$\begin{aligned} I_{R_1} &= I_1 + I_2 \\ &= 0.7 \text{ A} \end{aligned}$$

The voltage across R_1 is V_A which is obtained applying Ohm's law:

$$\begin{aligned} V_A &= I_{R_1} R_1 \\ &= 70 \text{ V} \end{aligned}$$

- *Circuit b):* For this circuit we can write the following set of eqns:

$$\begin{cases} I_{R_1} = I_{R_2} \\ V_A - V_C = V_1 \\ V_B = V_2 \end{cases} \tag{1.2}$$

which can be rewritten as follows:

$$\begin{cases} \frac{V_B - V_A}{R_1} = \frac{V_C}{R_2} \\ V_A - V_C = V_1 \\ V_B = V_2 \end{cases} \tag{1.3}$$

Solving this set of eqns in order to obtain V_A, V_B and V_C we can write:

$$\begin{aligned} V_A &= \frac{R_2 V_2 + R_1 V_1}{R_1 + R_2} \\ &= 2.63 \text{ V} \\ V_B &= V_2 \\ &= 3 \text{ V} \\ V_C &= R_2 \frac{V_2 - V_1}{R_1 + R_2} \\ &= 0.63 \text{ V} \end{aligned}$$

The current I_{R_1} is equal to $I_{R_2} = 3.7$ mA.

- *Circuit c):* For this circuit we can write

$$\begin{cases} I_{R_1} = \frac{V_A}{R_1} \\ V_A = V_1 \\ I_{R_2} = I_{R_3} \end{cases} \tag{1.4}$$

that is,

$$\begin{cases} I_{R_1} = \frac{V_1}{R_1} \\ \frac{V_1 - V_B}{R_2} = \frac{V_B}{R_3} \end{cases} \tag{1.5}$$

Solving to obtain V_B we get

$$\begin{aligned} V_B &= V_1 \frac{R_3}{R_2 + R_3} \\ &= 1.1 \text{ V} \end{aligned}$$

The voltage across R_2 is $V_A - V_B = 0.9$ V. The currents through the resistances are:

$$\begin{aligned} I_{R_1} &= 4 \text{ mA} \\ I_{R_2} &= I_{R_3} = 7.5 \text{ mA} \end{aligned}$$

- *Circuit d):* For this circuit we write the following set of eqns:

$$\begin{cases} I_{R_1} + I_{R_2} + I_{R_3} = 0 \\ I_{R_4} = I_{R_3} \\ V_B - V_A = V_1 \\ V_B - V_C = V_2 \end{cases} \tag{1.6}$$

This set of eqns can be written as:

$$\begin{cases} \frac{V_A}{R_1} + \frac{V_B}{R_2} + \frac{V_D}{R_3} = 0 \\ \frac{V_C - V_D}{R_4} = \frac{V_D}{R_3} \\ V_B - V_A = V_1 \\ V_B - V_C = V_2 \end{cases} \tag{1.7}$$

Solving to obtain V_A, V_B, V_C and V_D we get:

$$\begin{aligned} V_A &= R_1 \frac{R_2 V_2 - V_1(R_2 + R_3 + R_4)}{R_2(R_1 + R_3 + R_4) + R_1(R_3 + R_4)} \\ &= -29.3 \ \text{mV} \\ V_B &= R_2 \frac{R_1 V_2 + V_1(R_3 + R_4)}{R_2(R_1 + R_3 + R_4) + R_1(R_3 + R_4)} \\ &= 1.97 \ \text{V} \\ V_C &= \frac{V_1 R_2(R_3 + R_4) - V_2(R_2 R_3 + R_2 R_4 + R_1 + R_3 + R_1 R_4)}{R_2(R_1 R_3 + R_4) + R_1(R_3 + R_4)} \\ &= -1.03 \ \text{V} \\ V_D &= -R_3 \frac{(R_1 + R_2) V_2 - R_2 V_1}{R_2(R_1 + R_3 + R_4) + R_1(R_3 + R_4)} \\ &= -0.46 \ \text{V} \end{aligned}$$

- *Circuit e):* For this circuit we write

$$\begin{cases} I_{R_4} = I_{R_1} + I_{R_3} \\ I_{R_2} = I_{R_3} \\ V_A = V_1 \end{cases} \tag{1.8}$$

or

$$\begin{cases} \frac{V_C}{R_4} = \frac{V_1 - V_C}{R_1} + \frac{V_B - V_C}{R_3} \\ \frac{V_1 - V_B}{R_2} = \frac{V_B - V_C}{R_3} \end{cases} \tag{1.9}$$

Solving to obtain V_B and V_C:

$$\begin{aligned} V_B &= V_1 \frac{R_1 R_3 + R_4(R_1 + R_2 + R_3)}{R_1(R_2 + R_3 + R_4) + R_4(R_2 + R_3)} \\ &= 1.8 \ \text{V} \\ V_C &= V_1 \frac{R_4(R_1 + R_2 + R_3)}{R_1(R_2 + R_3 + R_4) + R_4(R_2 + R_3)} \\ &= 1.33 \ \text{V} \end{aligned}$$

- *Circuit f):* For this circuit we write the following set of eqns:

$$\begin{cases} I_1 = I_{R_1} + I_{R_3} \\ I_1 = I_{R_2} + I_{R_4} \\ V_1 = V_A - V_C \end{cases} \tag{1.10}$$

that is

$$\begin{cases} I_1 = \frac{V_B - V_A}{R_1} + \frac{V_B - V_C}{R_3} \\ I_1 = \frac{V_A}{R_2} + \frac{V_C}{R_4} \\ V_1 = V_A - V_C \end{cases} \tag{1.11}$$

Solving to obtain V_A, V_B and V_C we get

$$\begin{aligned} V_A &= R_2 \frac{V_1 + R_4 I_1}{R_2 + R_4} \\ &= 8.75 \;\; \text{V} \\ V_B &= V_1 \frac{R_2 R_3 - R_1 R_4}{(R_2 + R_4)(R_1 + R_3)} + I_1 \left[\frac{R_1 R_3}{R_1 + R_3} + \frac{R_2 R_4}{R_2 + R_4} \right] \\ &= 14.46 \;\; \text{V} \\ V_C &= R_4 \frac{R_2 I_1 - V_1}{R_2 + R_4} \\ &= 6.75 \;\; \text{V} \end{aligned}$$

Solution of problem 1.4

We consider N resistances connected in series and driven by a DC voltage source V as shown in figure 1.5. For this circuit we can write:

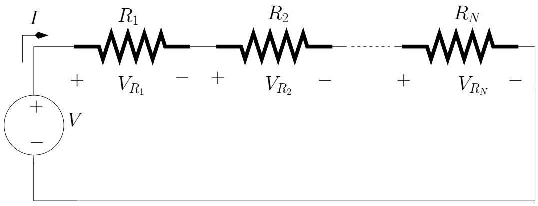


Figure 1.5: N *resistances connected in series.*

$$
\begin{aligned}
V &= V_{R_1} + V_{R_2} + \ldots + V_{R_N} \\
&= I\,(R_1 + R_2 + \ldots + R_N) \\
&= I\,R_{eq}
\end{aligned}
$$

that is

$$
R_{eq} = R_1 + R_2 + \ldots + R_N
$$

Solution of problem 1.5

We consider N resistances connected in parallel and driven by a DC voltage source V as shown in figure 1.6. For this circuit we can write:

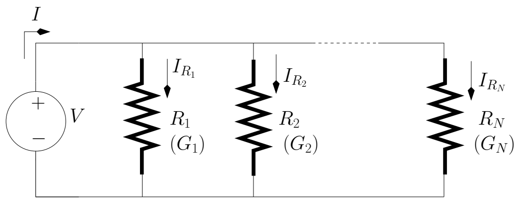


Figure 1.6: N *resistances connected in parallel.*

$$\begin{aligned} I &= I_{R_1} + I_{R_2} + \ldots + I_{R_N} \\ &= V\,(G_1 + G_2 + \ldots + G_N) \\ &= V\,G_{eq} \end{aligned}$$

that is

$$G_{eq} = G_1 + G_2 + \ldots + G_N$$

Solution of problem 1.6

- *Circuit a)*: The equivalent resistance between points A and B is given by the parallel combination of the three resistance, that is:

  $$R_{eq} = (R_1||R_2)||R_3$$

  Let $R_{1,2}$ be the resistance resulting from the parallel combination of R_1 and R_2;

  $$\begin{aligned} R_{1,2} &= \frac{R_1R_2}{R_1+R_2} \\ &= 22.2 \ \Omega \end{aligned}$$

  R_{eq} can be calculated as follows:

  $$\begin{aligned} R_{eq} &= \frac{R_{1,2}R_3}{R_{1,2}+R_3} \\ &= 16.2 \ \Omega \end{aligned}$$

  The equivalent conductance is $G_{eq} = R_{eq}^{-1} = 61.7$ mS.

- *Circuit b)*: Since the three resistances are connected in series we can write

  $$\begin{aligned} R_{eq} &= R_1+R_2+R_3 \\ &= 210 \ \Omega \end{aligned}$$

  The equivalent conductance is $G_{eq} = R_{eq}^{-1} = 4.8$ mS

- *Circuit c)*: The resistance R_1 is connected in series with R_2;

  $$\begin{aligned} R_{1,2} &= R_1+R_2 \\ &= 400 \ \Omega \end{aligned}$$

  $R_{1,2}$ is connected in parallel with R_3:

  $$\begin{aligned} R_{1,2,3} &= \frac{R_{1,2}R_3}{R_{1,2}+R_3} \\ &= 114.3 \ \Omega \end{aligned}$$

  $R_{1,2,3}$ is connected in series with R_4, that is

  $$\begin{aligned} R_{eq} &= R_{1,2,3}+R_4 \\ &= 384.3 \ \Omega \end{aligned}$$

  The equivalent conductance is $G_{eq} = R_{eq}^{-1} = 2.6$ mS.

- *Circuit d)*: The resistance R_1 is connected in series with R_2 and R_3 is connected in series with R_4;

  $$\begin{aligned} R_{1,2} &= R_1+R_2 \\ &= 150 \ \Omega \\ R_{3,4} &= R_3+R_4 \\ &= 150 \ \Omega \end{aligned}$$

  The equivalent resistance results from the parallel combination of $R_{1,2}$ with R_5 and with $R_{3,4}$

  $$\begin{aligned} R_{eq} &= R_{1,2}||R_5||R_{3,4} \\ &= 42.9 \ \Omega \end{aligned}$$

  The equivalent conductance is $G_{eq} = R_{eq}^{-1} = 23.3$ mS.

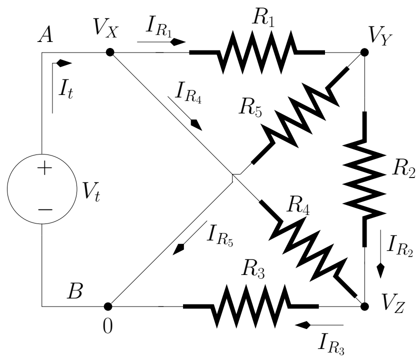


Figure 1.7: *Circuit of problem 2 e).*

- *Circuit e)* For this circuit we cannot identify any combination of resistances which share the same voltage across their terminals or share the same current. This means that there are no parallel nor series connections. In order to determine the equivalent resistance we have to apply a test voltage source V_t to the circuit between points A and B as shown in figure 1.7. This source supplies a current I_t to the circuit. By evaluating V_t/I_t we calculate the equivalent resistance:

$$R_{eq} = \frac{V_t}{I_t} \tag{1.12}$$

We apply the Nodal analysis method to evaluate V_t/I_t. For the circuit of figure 1.7 we can write

$$\begin{cases} I_t = I_{R_1} + I_{R_4} \\ I_t = I_{R_5} + I_{R_3} \\ I_{R_1} = I_{R_5} + I_{R_2} \\ V_X = V_t \end{cases} \tag{1.13}$$

that is,

$$\begin{cases} I_t = \dfrac{V_t - V_Y}{R_1} + \dfrac{V_t - V_Z}{R_4} \\ \\ I_t = \dfrac{V_Y}{R_5} + \dfrac{V_Z}{R_3} \\ \\ \dfrac{V_t - V_Y}{R_1} = \dfrac{V_Y}{R_5} + \dfrac{V_Y - V_Z}{R_2} \end{cases} \tag{1.14}$$

Solving in order to obtain V_t we obtain:

$$V_t = I_t \frac{R_5R_3(R_1 + R_2 + R_4) + R_1R_4(R_2 + R_3 + R_5) + R_2(R_4R_5 + R_1R_3)}{R_2(R_1 + R_3 + R_4 + R_5) + (R_3 + R_5)(R_1 + R_4)}$$

If we divide both terms of last eqn by I_t we obtain the desired equivalent resistance;

$$\begin{aligned} \frac{V_t}{I_t} &= \frac{R_5R_3(R_1 + R_2 + R_4) + R_1R_4(R_2 + R_3 + R_5) + R_2(R_4R_5 + R_1R_3)}{R_2(R_1 + R_3 + R_4 + R_5) + (R_3 + R_5)(R_1 + R_4)} \\ &= 83.5\ \Omega \end{aligned}$$

The equivalent conductance is $G_{eq} = R_{eq}^{-1} = 12.0$ mS.

Solution of problem 1.7

We consider N capacitances connected in series and driven by a voltage source $v(t)$ as shown in figure 1.8. For this circuit we can write:

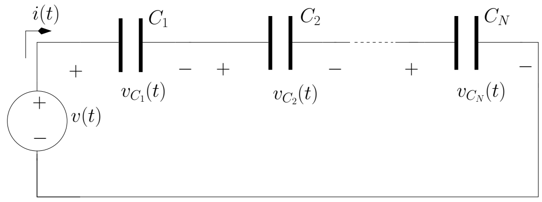


Figure 1.8: *N capacitances connected in series.*

$$
\begin{aligned}
v(t) &= v_{C_1}(t) + v_{C_2}(t) + \ldots + v_{C_N}(t) \\
&= \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}\right) \int_0^t i(t)dt \\
&= \frac{1}{C_{eq}} \int_0^t i(t)dt
\end{aligned}
$$

that is

$$
\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}
$$

Solution of problem 1.8

We consider N capacitances connected in parallel and driven by a voltage source $v(t)$ as shown in figure 1.9. For this circuit we can write:

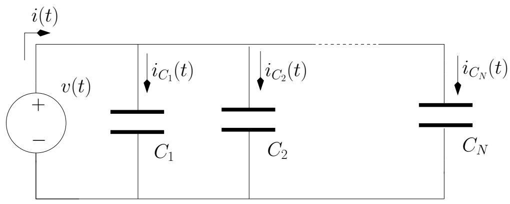


Figure 1.9: *N capacitances connected in parallel.*

$$
\begin{aligned}
i(t) &= i_{C_1}(t) + i_{C_2}(t) + \ldots + i_{C_N}(t) \\
&= (C_1 + C_2 + \ldots + C_N)\,\frac{d\,v(t)}{dt} \\
&= C_{eq}\frac{d\,v(t)}{dt}
\end{aligned}
$$

that is

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

Solution of problem 1.9

- *Circuit a)*: All capacitors are connected in series:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

  Solving, we obtain:

$$C_{eq} = 0.67 \;\mu\text{F}$$

- *Circuit b)*: Now, all capacitors are connected in parallel:

$$\begin{aligned} C_{eq} &= C_1 + C_2 + C_3 \\ &= 11 \;\mu\text{F} \end{aligned}$$

- *Circuit c)*: C_1 is connected in parallel with C_2. The capacitance resulting from this combination is connected in series with C_3;

$$\begin{aligned} C_{eq} &= \frac{(C_1 + C_2)C_3}{(C_1 + C_2) + C_3} \\ &= 3 \;\mu\text{F} \end{aligned}$$

Solution of problem 1.10

We consider N inductances connected in series and driven by a voltage source $v(t)$ as shown in figure 1.10. For this circuit we can write:

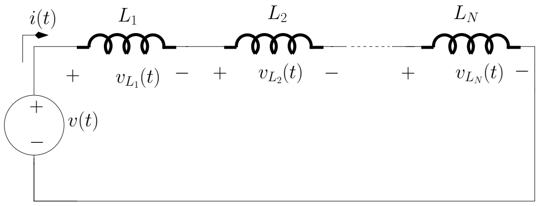


Figure 1.10: N *inductances connected in series.*

$$\begin{aligned} v(t) &= v_{L_1}(t) + v_{L_2}(t) + \ldots + v_{L_N}(t) \\ &= (L_1 + L_2 + \ldots + L_N)\,\frac{d\,i(t)}{dt} \\ &= L_{eq}\frac{d\,i(t)}{dt} \end{aligned}$$

that is

$$L_{eq} = L_1 + L_2 + \ldots + L_N$$

Solution of problem 1.11

We consider N inductances connected in parallel and driven by a voltage source $v(t)$ as shown in figure 1.11. For this circuit we can write:

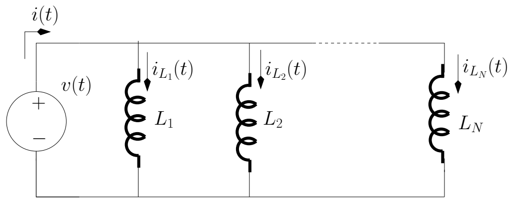


Figure 1.11: N *inductances connected in parallel.*

$$
\begin{aligned}
i(t) &= i_{L_1}(t) + i_{L_2}(t) + \ldots + i_{L_N}(t) \\
&= \left(\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}\right) \int_0^t v(t)\, dt \\
&= \frac{1}{L_{eq}} \int_0^t v(t)\, dt
\end{aligned}
$$

that is

$$
\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}
$$

Solution of problem 1.12

- *Circuit a):* The inductance L_2 is connected in series with L_3 and L_5 is connected in parallel with L_6. Hence, we can write:

$$\begin{aligned} L_{2,3} &= L_2 + L_3 \\ &= 11 \ \text{mH} \\ L_{5,6} &= \frac{L_5\,L_6}{L_5 + L_6} \\ &= 5.74 \ \text{mH} \end{aligned}$$

The circuit of figure 1.12 a) can be simplified as shown in figure 1.12 b). From the circuit of

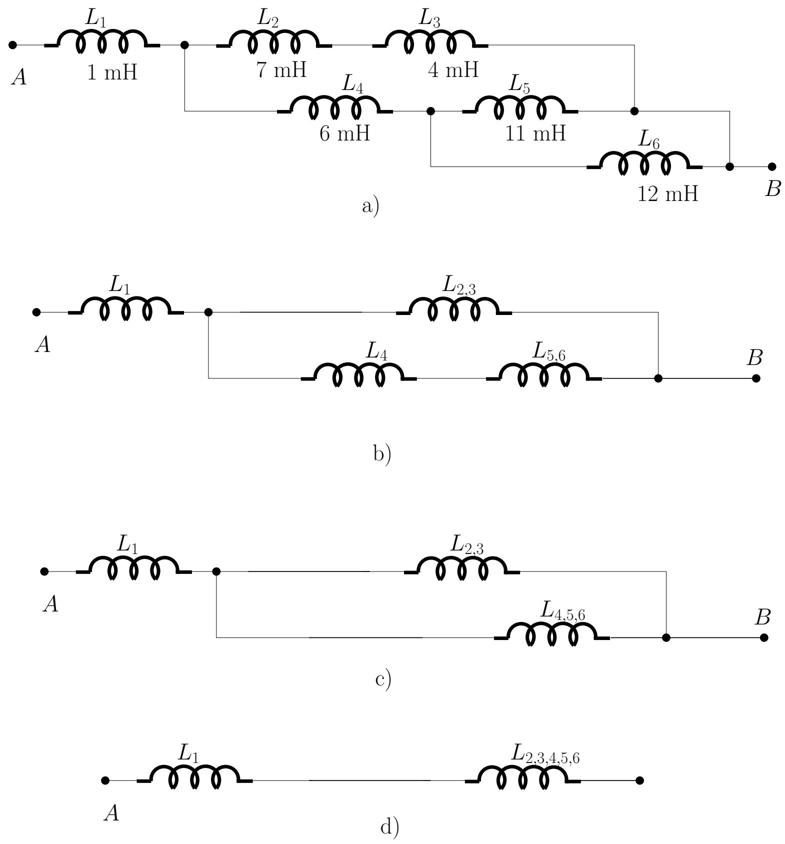


Figure 1.12: *Circuits of problem 1.12 a)*

figure 1.12 b) we can observe that L_4 is connected in series with $L_{5,6}$;

$$\begin{aligned} L_{4,5,6} &= L_4 + L_{5,6} \\ &= 11.74 \ \text{mH} \end{aligned}$$

From figure 1.12 c) we see that $L_{4,5,6}$ is connected in parallel with $L_{2,3}$;

$$\begin{aligned} L_{2,3,4,5,6} &= \frac{L_{4,5,6}L_{2,3}}{L_{4,5,6} + L_{2,3}} \\ &= 5.68 \ \text{mH} \end{aligned}$$

From figure 1.12 d) we see that the equivalent inductance is given by the series combination of L_1 with $L_{2,3,4,5,6}$, that is:

$$\begin{aligned} L_{eq} &= L_1 + L_{2,3,4,5,6} \\ &= 6.68 \ \text{mH} \end{aligned}$$

- *Circuit b)*: L_2 is connected in parallel with L_3. The resulting inductance is connected in series with L_1 and L_4;

$$
\begin{aligned}
L_{eq} &= L_1 + \frac{L_2 L_3}{L_2 + L_3} + L_4 \\
&= 25.26 \ \text{mH}
\end{aligned}
$$

Solution of problem 1.13

- *Circuit a):* Using the expression for the resistive current divider we can obtain the current through R_o as follows:

$$\begin{aligned} I_o &= 0.1\frac{R_1||R_2}{R_o + (R_1||R_2)} \\ &= 23.4 \ \text{mA} \end{aligned}$$

  and the voltage across R_o is 1.4 Volt.

- *Circuit b):* Using the expression for the resistive voltage divider we can obtain the voltage across R_o as follows:

$$\begin{aligned} V_o &= 3\,\frac{R_o}{R_o + R_1 + R_2} \\ &= 0.4 \ \text{V} \end{aligned}$$

  and the current through R_o is 10 mA.

Solution of problem 1.14

We use the Nodal Analysis method to analyse the circuits of figure 1.13.

- *Circuit a):* For this circuit we observe that the voltage across R_1 is the voltage supplied by V_s, that is, 5 V. Hence we have:

$$\begin{aligned} I_{R_1} &= \frac{V_s}{R_1} \\ &= 73.5 \ \text{mA} \end{aligned}$$

  Note that the current that flows through the short-circuit is $I_{sc} = I_{R_1} + I_s = 273.5$ mA.

- *Circuit b):* For the circuit of figure 1.13 b) we can write the following set of eqns:

$$\begin{cases} I_s = I_{R_1} + I_{R_3} + I_{R_5} \\ I_s = I_{R_2} + I_{R_4} + I_{R_5} \\ I_{R_1} = I_{R_2} \end{cases} \tag{1.15}$$

  that is

$$\begin{cases} I_s = \dfrac{V_A - V_B}{R_1} + \dfrac{V_A - V_C}{R_3} + \dfrac{V_A}{R_5} \\[2ex] I_s = \dfrac{V_B}{R_2} + \dfrac{V_C}{R_4} + \dfrac{V_A}{R_5} \\[2ex] \dfrac{V_A - V_B}{R_1} = \dfrac{V_B}{R_2} \end{cases} \tag{1.16}$$

  Solving in order to obtain V_A, V_B and V_C we get:

$$\begin{aligned} V_A &= I_s \frac{R_5(R_3+R_4)(R_2+R_1)}{R_5(R_1+R_2+R_3+R_4)+(R_4+R_3)(R_1+R_2)} \\ &= 30 \ \text{V} \\ V_B &= I_s \frac{R_5 R_2(R_3+R_4)}{R_5(R_1+R_2+R_3+R_4)+(R_4+R_3)(R_1+R_2)} \\ &= 14 \ \text{V} \\ V_C &= I_s \frac{R_5 R_4(R_2+R_1)}{R_5(R_1+R_2+R_3+R_4)+(R_4+R_3)(R_1+R_2)} \\ &= 18 \ \text{V} \end{aligned}$$

  V_{R_1} is equal to $(V_A - V_B) = 16$ V and $I_{R_1} = 200$ mA.

- *Circuit c):* For this circuit we can write:

$$\begin{cases} V_C - V_D = V_s \\ I_3 + I_2 + I_4 + I_6 = I_s \\ I_1 + I_s = I_5 \\ I_5 = I_4 + I_6 \end{cases} \tag{1.17}$$

  that is

$$\begin{cases} V_C - V_D = V_s \\[2ex] \dfrac{V_D}{R_3} + \dfrac{V_C}{R_2} + \dfrac{V_B}{R_4} + \dfrac{V_B}{R_6} = I_s \\[2ex] \dfrac{V_C - V_A}{R_1} + I_s = \dfrac{V_A - V_B}{R_5} \\[2ex] \dfrac{V_A - V_B}{R_5} = \dfrac{V_B}{R_4} + \dfrac{V_B}{R_6} \end{cases} \tag{1.18}$$

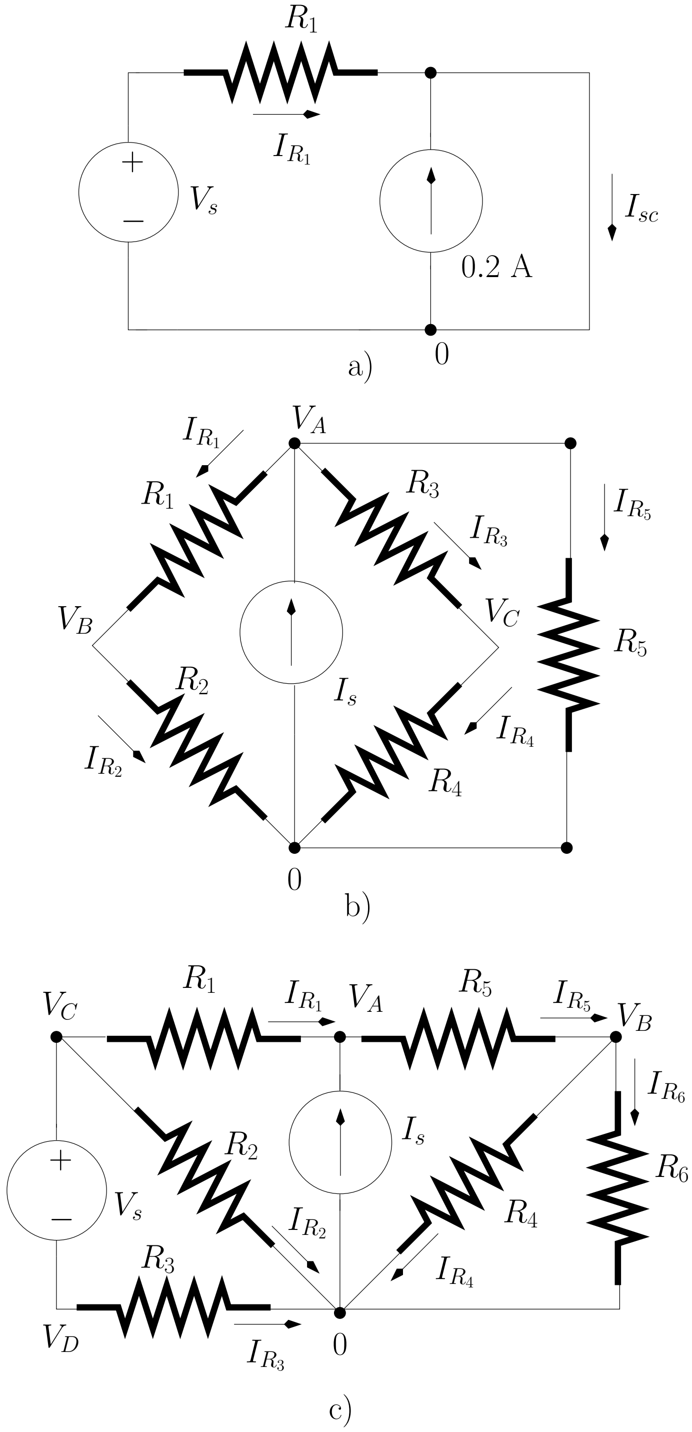


Figure 1.13: *Circuits of problem 1.14.*

Since R_4 is connected in parallel with R_6 the last set of eqns can be written as

$$\begin{cases} V_C - V_D = V_s \\ \dfrac{V_D}{R_3} + \dfrac{V_C}{R_2} + \dfrac{V_B}{R_{4,6}} = I_s \\ \dfrac{V_C - V_A}{R_1} + I_s = \dfrac{V_A - V_B}{R_5} \\ \dfrac{V_A - V_B}{R_5} = \dfrac{V_B}{R_{4,6}} \end{cases} \tag{1.19}$$

where $R_{4,6} = R_4||R_6 = 21.7\ \Omega$. Solving in order to obtain V_A, V_B, V_C and V_D we get

$$\begin{aligned} V_A &= \frac{R_4(R_{4,6}+R_5)[I_s(R_3R_1+R_2R_1+R_3R_2)+R_2V_s]}{(R_3R_4+R_2R_4)(R_{4,6}+R_5+R_1)+R_3R_2(R_{4,6}+R_4)} \\ &= 6.95\ \text{V} \\ V_B &= \frac{R_4R_{4,6}[I_s(R_3R_1+R_2R_1+R_3R_2)+R_2V_s]}{(R_3R_4+R_2R_4)(R_{4,6}+R_5+R_1)+R_3R_2(R_{4,6}+R_4)} \\ &= 2.71\ \text{V} \\ V_C &= \frac{R_2[V_sR_4(R_{4,6}+R_5+R_1)+I_sR_3(R_4R_{4,6}+R_4R_5-R_1R_{4,6})]}{(R_3R_4+R_2R_4)(R_{4,6}+R_5+R_1)+R_3R_2(R_{4,6}+R_4)} \\ &= 7.33\ \text{V} \\ V_D &= \frac{R_3I_sR_2[R_4(R_{4,6}+R_5)-R_1R_{4,6}]}{(R_3R_4+R_2R_4)(R_{4,6}+R_5+R_1)+R_3R_2(R_{4,6}+R_4)} \\ &- \frac{R_3V_s[R_4(R_{4,6}+R_5+R_1)+R_2(R_{4,6}+R_4)]}{(R_3R_4+R_2R_4)(R_{4,6}+R_5+R_1)+R_3R_2(R_{4,6}+R_4)} \\ &= -2.67\ \text{V} \end{aligned}$$

V_{R_1} is equal to $V_C - V_A = 0.37$ V and $I_{R_1} = 24.8$ mA.

Solution of problem 1.15

Thévenin equivalent circuits between points A and B (see figure 1.14).

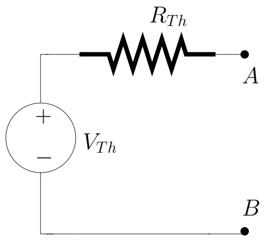


Figure 1.14: *Thévenin equivalent circuit between points A and B.*

- *Circuit b):* The Thévenin voltage, V_{Th}, is the voltage between points A and B obtained in the previous problem:

$$\begin{aligned} V_{Th} &= V_A \\ &= 30 \text{ V} \end{aligned}$$

The Thévenin resistance, R_{Th} is determined by calculating the equivalent resistance between the points A and B (after substituting the current source by an open-circuit). Figure 1.15 shows the equivalent circuit for the calculation of R_{Th}. From this figure we can write:

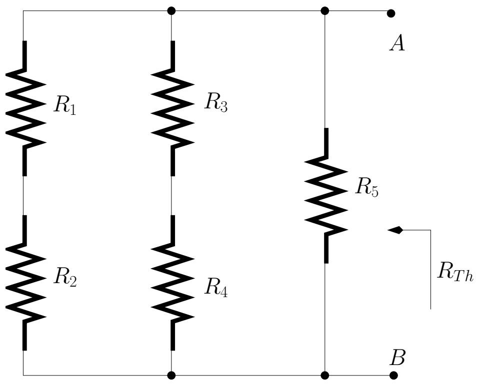


Figure 1.15: *Equivalent circuit for the calculation of R_{Th}.*

$$\begin{aligned} R_{Th} &= (R_1 + R_2)||(R_3 + R_4)||R_5 \\ &= 42.9 \ \Omega \end{aligned}$$

- *Circuit c):* The Thévenin voltage, V_{Th}, is the voltage between points A and B obtained in the previous problem:

$$\begin{aligned} V_{Th} &= V_A - V_B \\ &= 4.24 \text{ V} \end{aligned}$$

The Thévenin resistance, R_{Th} is determined by calculating the equivalent resistance between the points A and B (after substituting the current source by an open-circuit and the voltage source by a short-circuit). Figure 1.16 shows the equivalent circuit for the calculation of R_{Th}. From this figure we can write:

$$\begin{aligned} R_{Th} &= R_5||[(R2||R_3) + (R_4||R_6) + R_1] \\ &= 19.4 \ \Omega \end{aligned}$$

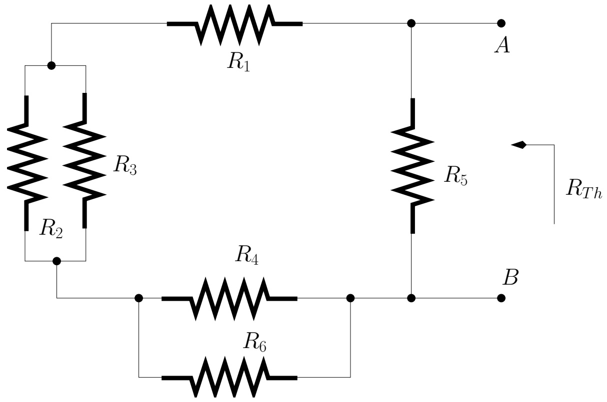


Figure 1.16: *Equivalent circuit for the calculation of R_{Th}.*

Solution of problem 1.16

Norton equivalent circuits between points A and B (see figure 1.17).

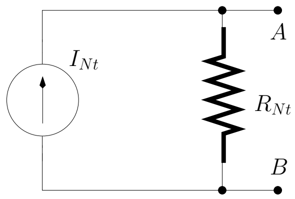


Figure 1.17: *Norton equivalent circuit.*

- *Circuit b):* The Norton current can be determined as follows:

$$\begin{aligned} I_{Nt} &= \frac{V_{Th}}{R_{Th}} \\ &= 699.3 \text{ mA} \end{aligned}$$

and the Norton resistance is equal to $R_{Nt} = R_{Th} = 42.9\ \Omega$.

- *Circuit c):* The Norton current can be determined as follows:

$$\begin{aligned} I_{Nt} &= \frac{V_{Th}}{R_{Th}} \\ &= 219.1 \text{ mA} \end{aligned}$$

and $R_{Nt} = R_{Th} = 19.4\ \Omega$.

Solution of problem 1.17

We use the Nodal analysis method to solve the circuits of figure 1.18.

- *Circuit a):* For this circuit we can write the following set of equations:

$$\begin{cases} I_s + I + I_{R_2} = 0 \\ I_{R_2} = A_i\, I \\ I = -\frac{V_A}{R_1} \\ I_{R_3} = -A_i\, I \end{cases} \tag{1.20}$$

that is

$$\begin{cases} I_s + I + \frac{V_B - V_A}{R_2} = 0 \\ \frac{V_B - V_A}{R_2} = A_i\, I \\ I = -\frac{V_A}{R_1} \\ \frac{V_C}{R_3} = -A_i\, I \end{cases} \tag{1.21}$$

Solving in order to obtain V_A, V_B and V_C we get:

$$\begin{aligned} V_A &= \frac{R_1\, I_s}{1 + A_i} \\ &= 1.73 \text{ V} \\ V_B &= \frac{I_s\,(R_1 - A_i\, R_2)}{1 + A_i} \\ &= -8.65 \text{ V} \\ V_C &= \frac{I_s\, A_i\, R_3}{1 + A_i} \\ &= 16.15 \text{ V} \end{aligned}$$

The voltage across R_3 is V_C. Thus, $I_{R_3} = 230.7$ mA.

- *Circuit b):* We observe that $V = V_B$. Hence we can write:

$$\begin{cases} \frac{V_C}{R_3} = G_m\, V \\ V = V_s \frac{R_2}{R_2 + R_1} \end{cases} \tag{1.22}$$

that is,

$$\begin{aligned} V_C &= R_3\, G_m\, V_s \frac{R_2}{R_2 + R_1} \\ &= 32.1 \text{ V} \end{aligned}$$

The voltage across R_3 is V_C. Thus, $I_{R_3} = 321$ mA.

- *Circuit c):* For this circuit we can write

$$\begin{cases} I_{R_1} = I_{R_2} + I_{R_4} \\ V_B - V_C = A_v V_B \\ I_{R_1} = I_{R_2} + I_{R_3} \end{cases} \tag{1.23}$$

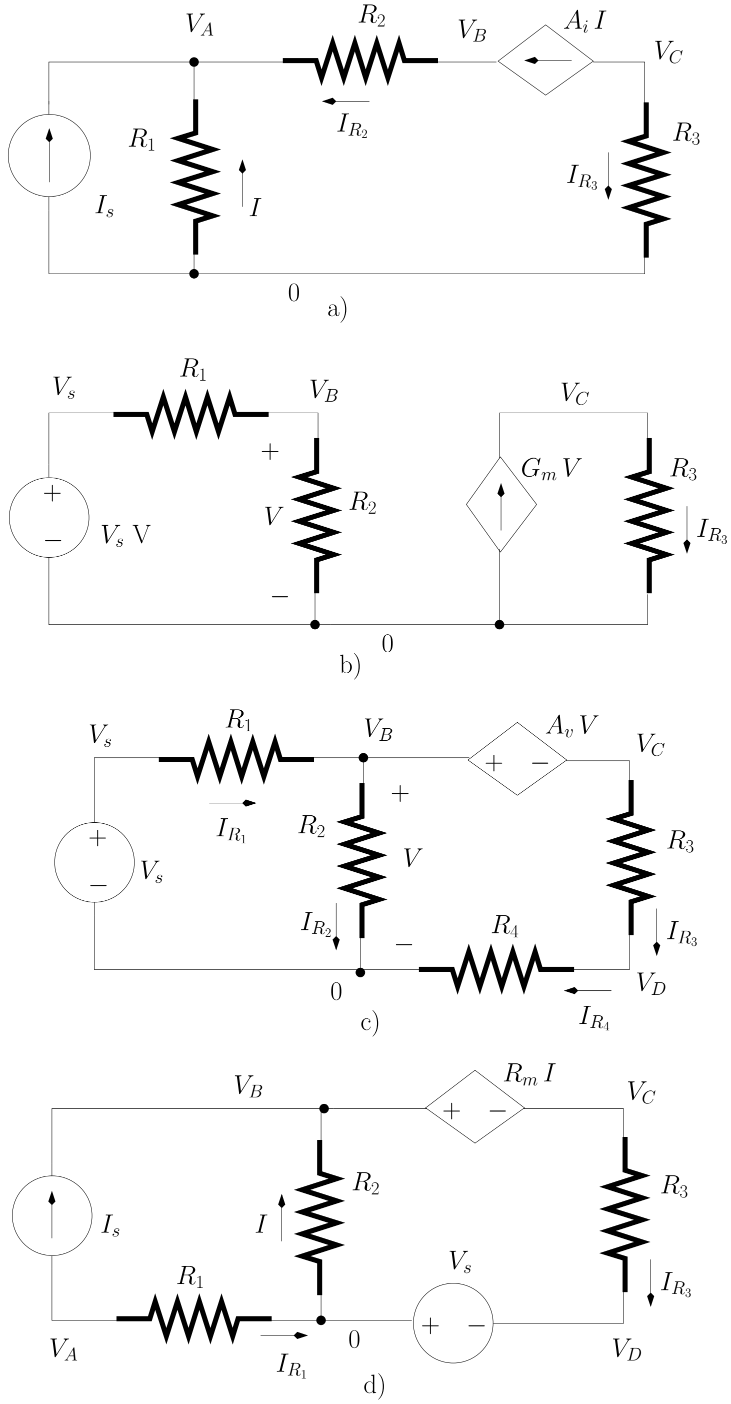


Figure 1.18: *Circuits of problem 1.17.*

that is

$$\begin{cases} \dfrac{V_s - V_B}{R_1} = \dfrac{V_B}{R_2} + \dfrac{V_D}{R_4} \\ V_B - V_C = A_v V_B \\ \dfrac{V_s - V_B}{R_1} = \dfrac{V_B}{R_2} + \dfrac{V_C - V_D}{R_3} \end{cases} \tag{1.24}$$

Solving to obtain V_B, V_C and V_D we have:

$$\begin{aligned} V_B &= \frac{V_s R_2 (R_3 + R_4)}{R_3(R_2 + R_1) + R_1 R_2 (1 - A_v) + R_4(R_1 + R_2)} \\ &= -0.56 \text{ V} \\ V_C &= \frac{-V_s R_2 (R_3 + R_4)(A_v - 1)}{R_3(R_2 + R_1) + R_1 R_2 (1 - A_v) + R_4(R_1 + R_2)} \\ &= 5.08 \text{ V} \\ V_D &= \frac{-R_4 V_s R_2 (A_v - 1)}{R_3(R_2 + R_1) + R_1 R_2 (1 - A_v) + R_4(R_1 + R_2)} \\ &= 4.32 \text{ V} \end{aligned}$$

The voltage across R_3 is $(V_C - V_D) = 0.76$ V and $I_{R_3} = 76$ mA.

- *Circuit d):* For this circuit we write:

$$\begin{cases} V_B - V_C = R_m \dfrac{-V_B}{R_2} \\ V_D = -V_s \\ \dfrac{V_A}{R_1} + \dfrac{V_C - V_D}{R_3} = \dfrac{-V_B}{R_2} \\ \dfrac{V_A}{R_1} = -I_s \end{cases} \tag{1.25}$$

Solving to obtain V_A, V_B, V_C and V_D we get:

$$\begin{aligned} V_A &= -I_s R_1 \\ &= -23.4 \text{ V} \\ V_B &= \frac{R_2 (I_s R_3 - V_s)}{R_2 + R_m + R_3} \\ &= 4.6 \text{ V} \\ V_C &= \frac{(I_s R_3 - V_s)(R_2 + R_m)}{R_2 + R_m + R_3} \\ &= 14.8 \text{ V} \\ V_D &= -V_s \\ &= -10 \text{ V} \end{aligned}$$

The voltage across R_3 is $V_C - V_D = -4.8$ V and $I_{R_3} = -66.7$ mA.

Solution of problem 1.18

We apply the Superposition theorem to the circuits of figure 1.19 to find the current flowing through R_2 (I_{R_2}).

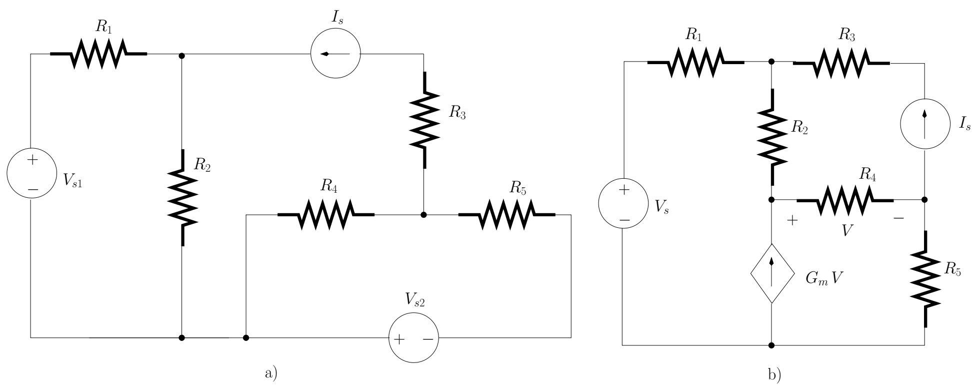


Figure 1.19: *Circuits of problem 1.18.*

1. *Circuit a)*

   - $\underline{V_{s1}}$: Figure 1.20 shows the equivalent circuit for the calculation of the contribution from V_{s1} to the current I_{R_2}. The current source I_s has been replaced by an open-circuit and the voltage source V_{s2} has been replaced by a short-circuit. From this circuit we can write:

$$\begin{aligned} I_{R_2} &= \frac{V_{s1}}{R_1 + R_2} \\ &= 40.5 \text{ mA} \end{aligned}$$

   - $\underline{I_s}$: Figure 1.21 shows the equivalent circuit for the calculation of the contribution from I_s to the current I_{R_2}. Both voltage sources were replaced by short-circuits. From this figure we observe that R_1 and R_2 form a resistive current divider. Hence we have:

$$\begin{aligned} I_{R_2} &= \frac{R_1}{R_1 + R_2} I_s \\ &= 64.9 \text{ mA} \end{aligned}$$

   - $\underline{V_{s2}}$: Figure 1.22 shows the equivalent circuit for the calculation of the contribution from V_{s2} to the current I_{R_2}. The current source I_s was replaced by an open-circuit and the voltage source V_{s1} was replaced by a short-circuit. We observe that there is no closed electrical path between R_2 and V_{s2}. Hence the contribution from V_{s2} to the current I_{R_2} is zero.The current that flows through R_2 (sum of all contributions) is $I_{R_2} = 40.5 + 64.9 = 105.4$ mA.

2. *Circuit b)*

   - $\underline{V_{s1}}$: Figure 1.23 shows the equivalent circuit for the calculation of the contribution from V_s to the current I_{R_2}. The current source I_s was replaced by an open-circuit. For this circuit we can write:

$$\begin{cases} I_{R_1} = I_{R_2} \\ I_{R_2} + G_m V = I_{R_4} \\ I_{R_4} = I_{R_5} \\ V = V_C - V_D \end{cases} \tag{1.26}$$

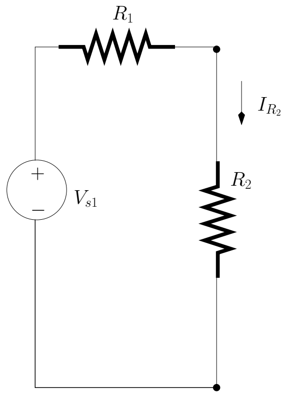


Figure 1.20: *Equivalent circuit for the calculation of the contribution from V_{s1} to the current I_{R_2}.*

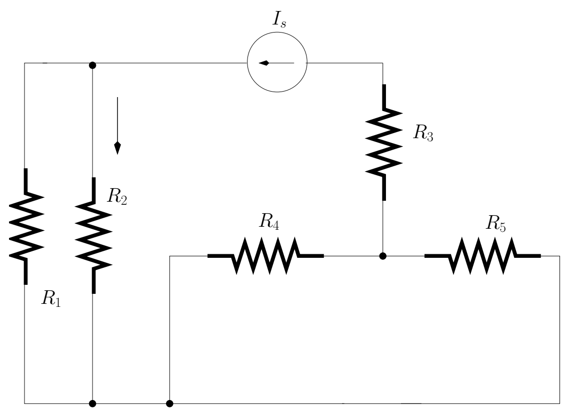


Figure 1.21: *Equivalent circuit for the calculation of the contribution from I_s to the current I_{R_2}.*

that is

$$\begin{cases} \dfrac{V_s - V_B}{R_1} = \dfrac{V_B - V_C}{R_2} \\ \\ \dfrac{V_B - V_C}{R_2} + G_m(V_C - V_D) = \dfrac{V_C - V_D}{R_4} \\ \\ \dfrac{V_C - V_D}{R_4} = \dfrac{V_D}{R_5} \end{cases} \tag{1.27}$$

Solving, we obtain:

$$\begin{aligned} V_B &= -V_s \frac{R_5 + R_4 - G_m R_2 R_4 + R_2}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \\ &= 1.79 \text{ V} \\ V_C &= -V_s \frac{R_5 + R_4}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \\ &= -0.08 \text{ V} \\ V_D &= -V_s \frac{R_5}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \\ &= -0.03 \text{ V} \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} I_{R_2} &= \frac{V_B - V_C}{R_2} \\ &= 20.8 \text{ mA} \end{aligned}$$

- $\underline{I_s}$: Figure 1.24 shows the equivalent circuit for the calculation of the contribution from I_s to the current I_{R_2}. The voltage source V_s was replaced by a short-circuit. For this

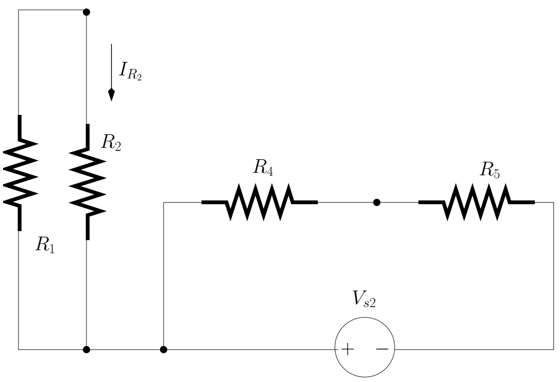


Figure 1.22: *Equivalent circuit for the calculation of the contribution from V_{s2} to the current I_{R_2}.*

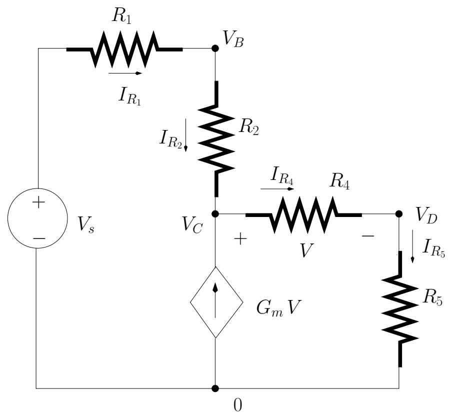


Figure 1.23: *Equivalent circuit for the calculation of the contribution from V_s to the current I_{R_2}.*

circuit we write:

$$\begin{cases} I_s = I_{R_3} \\ I_s = I_{R_1} + I_{R_2} \\ I_{R_2} + G_m V = I_{R_4} \\ V = (V_C - V_D) \\ I_{R_4} = I_{R_5} + I_s \end{cases} \tag{1.29}$$

that is

$$\begin{cases} I_s = \dfrac{V_A - V_B}{R_3} \\[2ex] I_s = \dfrac{V_B}{R_1} + \dfrac{V_B - V_C}{R_2} \\[2ex] \dfrac{V_B - V_C}{R_2} + G_m(V_C - V_D) = \dfrac{V_C - V_D}{R_4} \\[2ex] \dfrac{V_C - V_D}{R_4} = \dfrac{V_D}{R_5} + I_s \end{cases} \tag{1.30}$$

Solving in order to obtain V_A, V_B, V_C and V_D we get:

$$\begin{aligned} V_A &= I_s \frac{(G_m R_4 - 1)(R_3 R_2 + R_3 R_1 + R_1 R_2)}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \\ &- I_s \frac{R_3(R_5 + R_4) + R_1 R_4 (1 + G_m R_5)}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \\ &= 1.3 \text{ V} \\ V_B &= -I_s R_1 \frac{R_5 G_m R_4 + R_4 - G_m R_2 R_4 + R_2}{G_m R_2 R_4 - R_2 - R_5 - R_4 + R_1 G_m R_4 - R_1} \end{aligned}$$

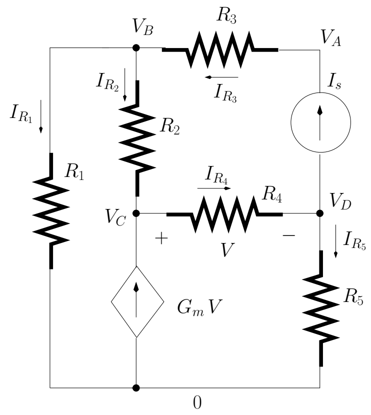


Figure 1.24: *Equivalent circuit for the calculation of the contribution from I_s to I_{R_2}.*

$$
\begin{aligned}
&= 0.3 \text{ V} \\
V_C &= -I_s \frac{-R_2R_5 + R_2R_5G_mR_4 + R_5G_mR_4R_1 + R_4R_1}{G_mR_2R_4 - R_2 - R_5 - R_4 + R_1G_mR_4 - R_1} \\
&= -5.96 \text{ V} \\
V_D &= I_sR_5 \frac{-R_2 + G_mR_2R_4 + R_1G_mR_4 - R_4}{G_mR_2R_4 - R_2 - R_5 - R_4 + R_1G_mR_4 - R_1} \\
&= -5.81 \text{ V}
\end{aligned}
$$

and the current flowing through R_2 is:

$$
\begin{aligned}
I_{R_2} &= \frac{V_B - V_C}{R_2} \\
&= 69.6 \text{ mA}
\end{aligned}
$$

The total current (the sum of all the contributions) is 90.4 mA.

3. *Circuit c)*

- $\underline{V_s}$: Figure 1.25 a) shows the equivalent circuit for the calculation of the contribution from V_s to V_{R_2}. From this figure we observe that the voltage across R_2 is:

$$
\begin{aligned}
V_{R_2} &= -A_v\, V_C + V_s \\
&= (1 - A_v)\, V_s \\
&= -90 \text{ V}
\end{aligned}
$$

  Note that $V_C = V_s$. $I_{R_2} = -0.45$A.

- $\underline{I_r}$: Figure 1.25 b) shows the equivalent circuit for the calculation of the contribution from I_r to V_{R_2}. Note that V_C is zero. Hence, the voltage-controlled voltage source is now effectively a short-circuit and the voltage-controlled current source is now an open-circuit. From this figure we observe that the voltage across R_2 is zero.

  The overall voltage across R_2 is -90 V and the overall current through R_2 is -0.45 A.

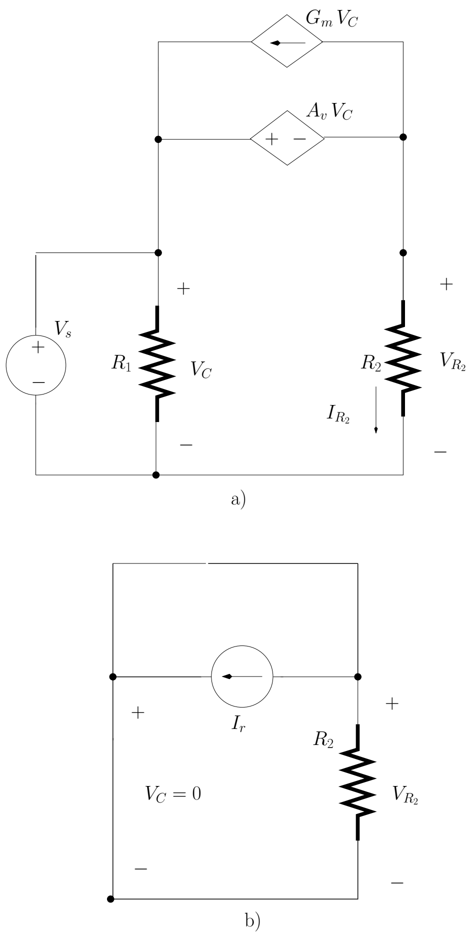


Figure 1.25: *a) Equivalent circuit for the calculation of the contribution from V_s to V_{R_2} b) Equivalent circuit for the calculation of the contribution from I_r to V_{R_2}.*

Chapter 2

Complex numbers: An introduction

Solution of problem 2.1

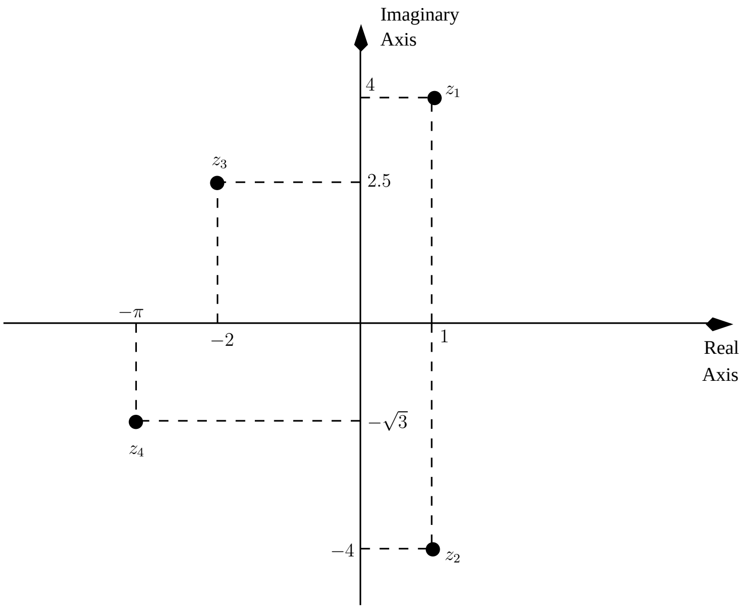


Figure 2.1: *Representation of the complex numbers of problem 1.*

Solution of problem 2.2

1. 2
2. $j\,2$
3. $11.25 + j\,2.5$
4. $0.3562 + j\,0.3836$
5. $(-1.25, 0.9)$
6. $(2.2, 5.1)$
7. $(-4, -15.5)$
8. $(1.1746, 0.4483)$

Solution of problem 2.3

1. $z = -3/2 + j\sqrt{127}/2$ or $z = -3/2 - j\sqrt{127}/2$
2. $z = 1/4 + \sqrt{21}/4$ or $z = 1/4 - \sqrt{21}/4$
3. $z = 1/4 + j\,1/4\sqrt{39}$ or $z = 1/4 - j\,1/4\sqrt{39}$
4. $z = -3$
5. $z = -\sqrt{7}/4 + j\sqrt{65}/4$ or $z = -\sqrt{7}/4 - j\sqrt{65}/4$

Solution of problem 2.4

1. $1.414\ \angle 0.785$ (rad)
2. $1.732\ \angle 2.186$ (rad)
3. $2.022\ \angle -0.149$ (rad)
4. $3.162\ \angle -2.562$ (rad)

Solution of problem 2.5

1. 0.5
2. $0.75 - j\,1.3$
3. $-j\,0.5$
4. $j\,0.5$

Solution of problem 2.6

1. 2
2. -8
3. $-13.753 + j\,9.992$
4. $0.072 + j\,0.222$
5. $4.511 + j\,8.142$
6. $0.45 - j\,0.279$

Solution of problem 2.7

1. $z = 1$ or $z = -1$
2. $z = \sqrt{2}/2(1 + j)$ or $z = \sqrt{2}/2(1 - j)$
3.
   - $z = 2^{1/6}\ \exp(j\pi/12)$ or
   - $z = 2^{1/6}\ \exp(j9\pi/12)$ or
   - $z = 2^{1/6}\ \exp(j17\pi/12)$
4.
   - $z = (5/2)^{1/5} \exp(-j3\pi/20)$ or
   - $z = (5/2)^{1/5} \exp(j5\pi/20)$ or
   - $z = (5/2)^{1/5} \exp(j13\pi/20)$ or
   - $z = (5/2)^{1/5} \exp(j21\pi/20)$ or
   - $z = (5/2)^{1/5} \exp(j29\pi/20)$.

Chapter 3

Frequency domain electrical signal and circuit analysis

Solution of problem 3.1

The effective value for each voltage waveform $v_i(t)$ can be determined as follows:

$$V_{eff_i} = \sqrt{\frac{1}{T}\int_{t_o}^{t_o+T} v_i^2(t)\,dt} \qquad (3.1)$$

where t_o can be chosen to facilitate the calculation of the integral in the last eqn.

- a) Using eqn 3.1 with $t_o = -T/2$ we can write:

  $$V_{eff_1} = \sqrt{\frac{1}{T}\int_{-T/2}^{T/2} v_1^2(t)\,dt}$$

  The integral of the last eqn can be determined as follows:

  $$\begin{aligned}
  \frac{1}{T}\int_{-T/2}^{T/2} v_1^2(t)\,dt &= \frac{V_A^2}{T}\int_{-T/4}^{T/4} \cos^2(\omega\,t)\,dt \ \text{ (with } \omega = 2\pi/T\text{)} \\
  &= \frac{V_A^2}{2\,T}\int_{-T/4}^{T/4} [1+\cos(2\,\omega\,t)]\,dt \\
  &= \frac{V_A^2}{2\,T}\Big[\ t\ \Big]_{-T/4}^{T/4} + \frac{V_A^2}{4\,T\,\omega}\Big[\sin(2\,\omega\,t)\Big]_{-T/4}^{T/4} \\
  &= \frac{V_A^2}{4}
  \end{aligned}$$

  Hence, $V_{eff_1} = V_A/2$.

- b) Using eqn 3.1 with $t_o = 0$ we can write:

  $$V_{eff_2} = \sqrt{\frac{1}{T}\int_{0}^{T} v_2^2(t)\,dt}$$

  The integral of the last eqn can be determined as follows:

  $$\begin{aligned}
  \frac{1}{T}\int_{0}^{T} v_2^2(t)\,dt &= \frac{V_B^2}{T^3}\int_{0}^{T} t^2\,dt \\
  &= \frac{V_B^2}{T^3}\left[\ \frac{t^3}{3}\ \right]_0^T \\
  &= \frac{V_B^2}{3}
  \end{aligned}$$

  Hence, $V_{eff_2} = V_B/\sqrt{3}$.

- c) Using eqn 3.1 with $t_o = -T/2$ we can write:

$$V_{eff_3} = \sqrt{\frac{1}{T}\int_{-T/2}^{T/2} v_3^2(t)\,dt}$$

The integral of the last eqn can be determined as follows:

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2} v_3^2(t)\,dt &= \frac{V_C^2}{T}\int_{-T/4}^{T/4} \sin^2(4\,\omega\,t)\,dt \;\; \text{(with } \omega = 2\pi/T\text{)} \\
&= \frac{V_C^2}{2\,T}\int_{-T/4}^{T/4} [1-\cos(8\,\omega\,t)]\,dt \\
&= \frac{V_C^2}{2\,T}\Big[\,t\,\Big]_{-T/4}^{T/4} - \frac{V_C^2}{2\,T\,8\,\omega}\Big[\sin(8\,\omega\,t)\Big]_{-T/4}^{T/4} \\
&= \frac{V_C^2}{4}
\end{aligned}$$

Hence, $V_{eff_3} = V_C/2$.

Solution of problem 3.2

Assuming that the AC voltage across Z_L is:

$$v_a(t) \quad = \quad V_a\, \cos(\omega\, t + \phi_v)$$

and that the AC current through Z_L is:

$$i_a(t) \quad = \quad I_a\, \cos(\omega\, t + \phi_i)$$

we can obtain the average power dissipated by Z_L as follows:

$$\begin{aligned}
P_{AV} &= \frac{1}{T}\int_0^T v_a(t)\, i_a(t)\, dt \\
&= \frac{V_a\, I_a}{2\,T}\int_0^T \cos(\phi_v - \phi_i)\, dt + \frac{V_a\, I_a}{2\,T}\int_0^T \cos(2\,\omega\, t + \phi_v + \phi_i)\, dt \\
&= \frac{V_a\, I_a}{2}\cos(\phi_v - \phi_i) \\
&= \frac{V_a\, I_a}{2}\text{Real}\left[e^{j\,\phi_v} \times e^{-j\,\phi_i}\right] \\
&= \frac{1}{2}\text{Real}[V_A\, I_A^*]
\end{aligned}$$

with

$$V_A = V_a\, e^{j\,\phi_v}$$
$$I_A = I_a\, e^{j\,\phi_i}$$

Solution of problem 3.3

We solve the circuits of this problem using the Nodal analysis method together with phasor analysis.

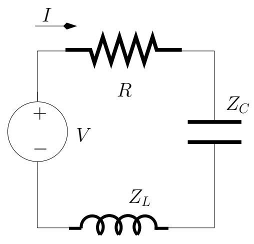


Figure 3.1: *Circuit (a) of problem 3.3.*

- *Circuit a):* The impedances associated with the capacitor, Z_C and the inductor, Z_L, are:

$$\begin{aligned} Z_C &= \left.\frac{1}{j\,\omega\,C}\right|_{\omega=30\ \text{krad/s}} \\ &= -j\,37.04\ \ \Omega \\ Z_L &= \left. j\,\omega\,L\right|_{\omega=30\ \text{krad/s}} \\ &= j\,90.0\ \ \Omega \end{aligned}$$

The (static) phasor associated with the voltage source is $V = 10\,\exp(j\,\pi/4)$ V. Since R is connected in series with C and with L we can determine the current I as follows:

$$\begin{aligned} I &= \frac{V}{Z_L + Z_C + R} \\ &= \frac{10\,e^{j\,\pi/4}}{j\,90 - j37.04 + 100} \\ &= 88.4\,e^{j\,0.30}\ \ \text{mA} \end{aligned}$$

The voltage across the resistance, V_R, can be obtained from:

$$\begin{aligned} V_R &= R\,I \\ &= 8.84\,e^{j\,0.30}\ \ \text{V} \end{aligned}$$

The voltage across the capacitance, V_C, can be obtained from:

$$\begin{aligned} V_C &= Z_C\,I \\ &= 3.27e^{j\,0.30 - j\,\pi/2}\ \ \text{V} \\ &= 3.27e^{-j\,1.27}\ \ \text{V} \end{aligned}$$

The voltage across the inductance, V_L, is:

$$\begin{aligned} V_L &= Z_L\,I \\ &= 7.95e^{j\,0.30 + j\,\pi/2}\ \ \text{V} \\ &= 7.95e^{j\,1.87}\ \ \text{V} \end{aligned}$$

Note that, as expected, the voltage and the current through the resistance are *in phase*, that is, the phase difference between the voltage and the current is zero. On the other hand, the voltage across the capacitor lags the current by $\pi/2$ while the voltage across the inductor leads the current by $\pi/2$.

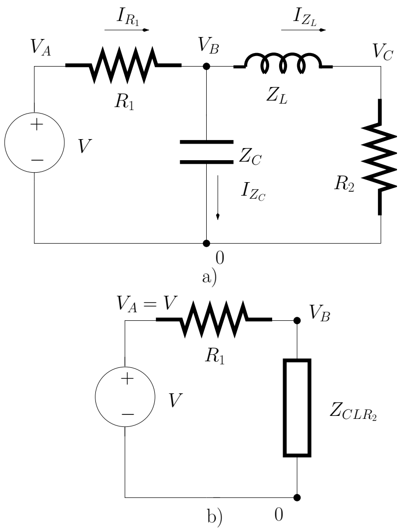


Figure 3.2: *a) Circuit (b) of figure 3.1. b) Equivalent circuit*

- *Circuit b):* The impedances associated with the capacitance and inductance are:

$$\begin{aligned} Z_C &= \left.\frac{1}{j\,\omega\,C}\right|_{\omega=30\ \text{krad/s}} \\ &= -j166.7\ \Omega \\ Z_L &= \left. j\,\omega\,L \right|_{\omega=30\ \text{krad/s}} \\ &= j\,300.0\ \Omega \end{aligned}$$

From figure 3.2 a) we observe that the series combination of R_2 with Z_L is connected in parallel with Z_C. Hence, we can obtain an equivalent impedance, which represent these combinations, as follows:

$$\begin{aligned} Z_{CLR_2} &= Z_C||(Z_L + R_2) \\ &= 77.3 - j\,201.0\ \Omega \end{aligned} \tag{3.2}$$

From figure 3.2 b) we recognise that R_1 and Z_{CLR_2} form an impedance voltage divider. Hence we can write:

$$\begin{aligned} V_B &= V\,\frac{Z_{CLR_2}}{Z_{CLR_2} + R_1} \\ &= 1.97\,e^{-j\,0.23}\ \text{V} \end{aligned}$$

From figure 3.2 a) we also note that Z_L and R_2 form an impedance voltage divider which allows us to relate V_C and V_B as follows:

$$\begin{aligned} V_C &= V_B\,\frac{R_2}{Z_L + R_2} \\ &= 1.39\,e^{-j\,1.02}\ \text{V} \end{aligned}$$

Now, the current in each element can be obtained:

$$\begin{aligned} I_{R_1} &= \frac{V - V_B}{R_1} \\ &= 9.1\,e^{j\,0.97}\ \text{mA} \\ I_{Z_C} &= \frac{V_B}{Z_C} \\ &= 11.8\,e^{j\,1.34}\ \text{mA} \end{aligned}$$

$$
\begin{aligned}
I_{Z_L} &= \frac{V_B - V_C}{Z_L} \\
&= 4.6\, e^{-j\,1.02} \ \text{mA} \\
I_{R_2} &= I_{Z_L}
\end{aligned}
$$

- *Circuit c):* The impedances associated with the capacitance and inductance are:

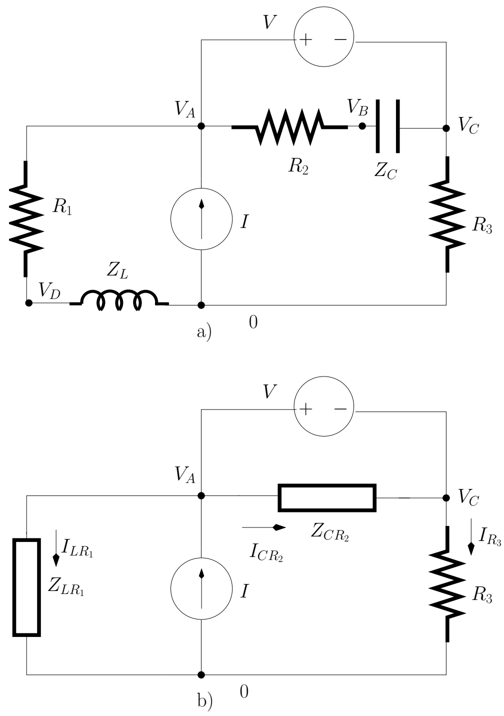


Figure 3.3: *a) Circuit (c) of figure 3.1. b) Equivalent circuit*

$$
\begin{aligned}
Z_C &= \left.\frac{1}{j\,\omega\,C}\right|_{\omega=30\ \text{krad/s}} \\
&= -j47.6 \ \ \Omega \\
Z_L &= \left. j\,\omega\,L\right|_{\omega=30\ \text{krad/s}} \\
&= j\,150 \ \ \Omega
\end{aligned}
$$

Since R_1 is in series with Z_L and R_2 is in series with Z_C we can obtain the equivalent circuit of figure 3.3 b) with

$$
\begin{aligned}
Z_{LR_1} &= Z_L + R_1 \\
&= 250 + j\,150 \ \ \Omega \\
Z_{CR_2} &= Z_C + R_2 \\
&= 200 - j47.6 \ \ \Omega
\end{aligned}
$$

For this circuit we can write the following set of eqns:

$$
\begin{cases}
I_{LR_1} + I_{R_3} = I \\
V_A - V_C = V
\end{cases}
\tag{3.3}
$$

that is:

$$
\begin{cases}
\dfrac{V_A}{Z_{LR_1}} + \dfrac{V_C}{R_3} = I \\
V_A - V_C = V
\end{cases}
\tag{3.4}
$$

Solving in order to obtain V_A and V_C we get:

$$\begin{aligned} V_A &= Z_{LR_1}\frac{V + I\,R_3}{R_3 + Z_{LR_1}} \\ &= 27.38\,e^{j\,1.26} \text{ V} \\ V_C &= R_3\frac{I\,Z_{LR_1} - V}{R_3 + Z_{LR_1}} \\ &= 19.05\,e^{j\,1.50} \text{ V} \end{aligned}$$

Hence, the current that flows through C and R_2 is given by:

$$\begin{aligned} I_{CR_2} &= \frac{V_A - V_C}{Z_{CR_2}} \\ &= 48.6\,e^{j\,1.02} \text{ mA} \end{aligned} \tag{3.5}$$

and the current that flows through L and R_1 is given by:

$$\begin{aligned} I_{LR_1} &= \frac{V_A}{Z_{LR_1}} \\ &= 93.9\,e^{j\,0.72} \text{ mA} \end{aligned} \tag{3.6}$$

The current that flows through R_3 is given by:

$$\begin{aligned} I_{R_3} &= \frac{V_C}{R_3} \\ &= 68.0\,e^{j\,1.50} \text{ mA} \end{aligned} \tag{3.7}$$

The voltages across R_2 and C are given by:

$$\begin{aligned} V_{R_2} &= I_{CR_2}\,R_2 \\ &= 9.73\,e^{j\,1.02} \text{ V} \\ V_{Z_C} &= I_{CR_2}\,Z_C \\ &= 2.32\,e^{-j\,0.55} \text{ V} \end{aligned}$$

The voltages across R_1 and L are given by:

$$\begin{aligned} V_{R_1} &= I_{LR_1}\,R_1 \\ &= 23.48\,e^{j\,0.72} \text{ V} \\ V_{Z_L} &= I_{LR_1}\,Z_L \\ &= 14.09\,e^{j\,2.29} \text{ V} \end{aligned}$$

- *Circuit d):* The impedances associated with the capacitance and inductance are:

$$\begin{aligned} Z_C &= \left.\frac{1}{j\,\omega\,C}\right|_{\omega=30 \text{ krad/s}} \\ &= -j\,33.3 \ \Omega \\ Z_L &= \left. j\,\omega\,L\right|_{\omega=30 \text{ krad/s}} \\ &= j\,180 \ \Omega \end{aligned}$$

Since R_1 is in parallel with Z_C and R_2 is in series with Z_L we can obtain the equivalent circuit of figure 3.4 b) with

$$\begin{aligned} Z_{CR_1} &= \frac{Z_C\,R_1}{Z_C + R_1} \\ &= 0.85 - j\,33.31 \ \Omega \\ Z_{LR_2} &= Z_L + R_2 \\ &= 800 + j180 \ \Omega \end{aligned}$$

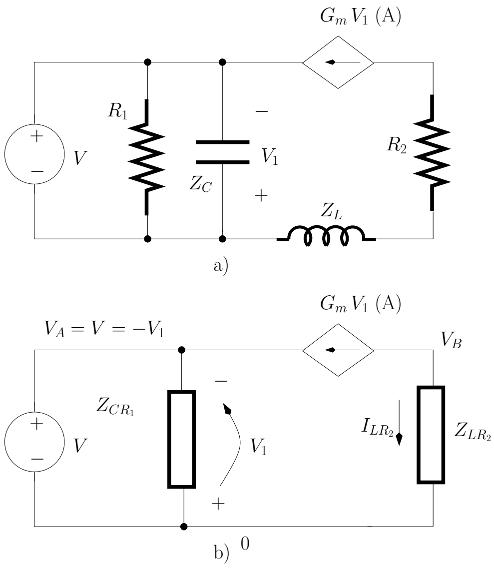


Figure 3.4: *a) Circuit (d) of figure 3.1. b) Equivalent circuit*

For this circuit we can write the following:

$$I_{LR_2} + G_m V_1 = 0$$

Since $V_1 = -V$ we can write the last eqn as:

$$\frac{V_B}{Z_{LR_2}} - G_m V = 0$$

that is

$$\begin{aligned} V_B &= Z_{LR_2} G_m V \\ &= 820.0\, e^{j\,1.01} \ \text{V} \end{aligned}$$

The currents that flow through C, I_{Z_C}, and through R_1, I_{R_1}, are given by

$$\begin{aligned} I_{Z_C} &= \frac{V}{Z_C} \\ &= 0.3\, e^{j\,2.36} \ \text{A} \\ I_{R_1} &= \frac{V}{R_1} \\ &= 7.7\, e^{j\,0.79} \ \text{mA} \end{aligned}$$

The current that flows through the series combination of L with R_2 is

$$I_{LR_2} = 1.0\, e^{j\,0.79} \ \text{A}$$

Solution of problem 3.4

In order to have maximum power transfer at $f = 35$ kHz Z_L must be equal to Z_S^* at this frequency. Z_S is the impedance of the series combination of the 120 Ω resistor and the capacitor (see figure

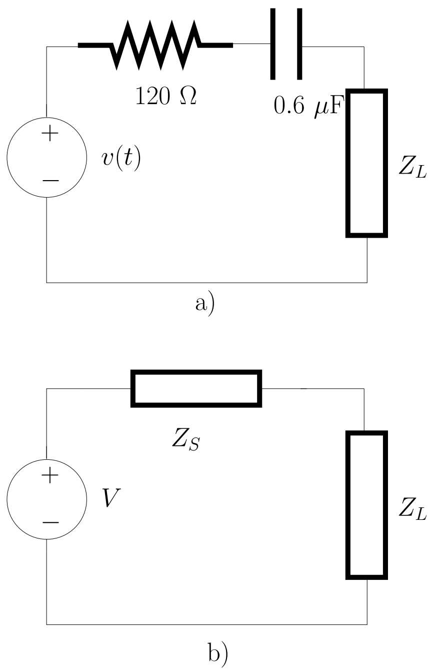


Figure 3.5: *a) Circuit of problem 3.4 b) Equivalent circuit.*

3.5):

$$
\begin{aligned}
Z_S &= R + \left.\frac{1}{j\,2\,\pi\,f\,C}\right|_{f=35\text{ kHz}} \\
&= 120 - j\,7.58\ \ \Omega
\end{aligned}
$$

Hence Z_L should be made equal to $Z_S^* = 120 + j\,7.58\ \ \Omega$. A circuit which realises this impedance $Z_L = Z_S^*$ at $f = 35$ kHz is the series combination of a 120 Ω resistance with an inductance L such that;

$$
\left. j\,2\,\pi\,f\,L \right|_{f=35\text{ kHz}} = j\,7.58
$$

that is $L = 34.46\ \mu$H.

Solution of problem 3.5

- a): The average value C_0 is determined as follows:

$$\begin{aligned} C_0 &= \frac{1}{T}\int_{-T/2}^{T/2} v_1(t)\,dt \\ &= \frac{1}{T}\int_{-T/4}^{T/4} V_A\,\cos(\omega\,t)\,dt \quad (\omega = 2\pi/T) \\ &= \frac{V_A}{T}\,\frac{T}{2\pi}\left[\sin\left(\frac{2\pi}{T}\,t\right)\right]_{-T/4}^{T/4} \\ &= \frac{V_A}{\pi} \end{aligned}$$

The coefficients C_n ($|n| > 0$) are determined as

$$\begin{aligned} C_n &= \frac{1}{T}\int_{-T/2}^{T/2} v_1(t)\,e^{-j\,2\,\pi\,\frac{n}{T}\,t}\,dt \\ &= \frac{V_A}{T}\int_{-T/2}^{T/2} \cos\left(2\,\pi\,\frac{1}{T}\,t\right)\,e^{-j\,2\,\pi\,\frac{n}{T}\,t}\,dt \\ &= \frac{V_A}{2\,T}\int_{-T/4}^{T/4} e^{j\,2\,\pi\,\frac{1-n}{T}\,t}\,dt + \frac{V_A}{2\,T}\int_{-T/4}^{T/4} e^{-j\,2\,\pi\,\frac{1+n}{T}\,t}\,dt \\ &= \frac{V_A}{2\,T}\,\frac{T}{j\,2\,\pi\,(1-n)}\,\left[e^{j\,2\,\pi\,\frac{1-n}{T}\,t}\right]_{-T/4}^{T/4} \\ &+ \frac{V_A}{2\,T}\,\frac{T}{-j\,2\,\pi\,(1+n)}\,\left[e^{-j\,2\,\pi\,\frac{1+n}{T}\,t}\right]_{-T/4}^{T/4} \\ &= \frac{V_A}{2\,\pi}\left(\frac{\sin\left[(1-n)\frac{\pi}{2}\right]}{1-n} + \frac{\sin\left[(1+n)\frac{\pi}{2}\right]}{1+n}\right) \\ &= \frac{V_A}{4}\text{sinc}\left(\frac{1-n}{2}\right) + \frac{V_A}{4}\text{sinc}\left(\frac{1+n}{2}\right) \end{aligned}$$

Figure 3.6 shows the spectrum of $v_1(t)$.

- b): The average value C_0 is determined as follows:

$$\begin{aligned} C_0 &= \frac{1}{T}\int_{0}^{T} v_2(t)\,dt \\ &= \frac{1}{T}\int_{0}^{T} \frac{V_B\,t}{T}\,dt \\ &= \frac{V_B}{2} \end{aligned} \tag{3.8}$$

The coefficients C_n ($|n| > 0$) are determined from

$$\begin{aligned} C_n &= \frac{1}{T}\int_{0}^{T} v_2(t)\,e^{-j\,2\,\pi\,\frac{n}{T}\,t}\,dt \\ &= \frac{1}{T}\int_{0}^{T} \frac{V_B\,t}{T}\,e^{-j\,2\,\pi\,\frac{n}{T}\,t}\,dt \\ &= \frac{V_B}{T^2}\left[\frac{-e^{-j\,2\,\pi\,\frac{n}{T}\,t}}{\left(-j\,2\,\pi\,\frac{n}{T}\right)^2}\left(1 + j\,2\,\pi\,\frac{n}{T}\,t\right)\right]_0^T \\ &= \frac{V_B}{(2\pi n)^2}\left[\underbrace{e^{-j\,2\,\pi\,n}}_{1}(1 + j\,2\,\pi\,n) - 1\right] \\ &= \frac{j\,V_B}{2\pi n} \end{aligned} \tag{3.9}$$

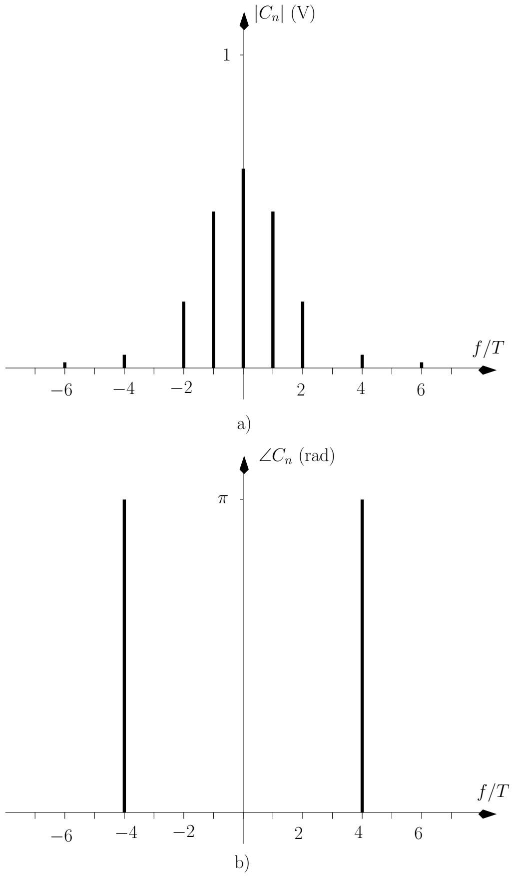


Figure 3.6: *Spectrum of $v_1(t)$ a) Magnitude. b) Phase.*

Figure 3.7 shows the spectrum of $v_2(t)$.

- c): The average value C_0 is zero. The coefficients C_n ($|n| > 0$) are determined from

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^T v_3(t)\, e^{-j\,2\,\pi\,\frac{n}{T}\,t}\, dt \\ &= \frac{1}{T}\int_{-T/4}^{T/4} V_C\, \cos(4\,\omega\,t)\, dt \quad (\omega = 2\pi/T) \\ &= \frac{V_C}{4}\text{sinc}\left(\frac{4-n}{2}\right) + \frac{V_A}{4}\text{sinc}\left(\frac{4+n}{2}\right) \end{aligned}$$

Figure 3.8 shows the spectrum of $v_3(t)$. Note that the spectrum contains no DC component.

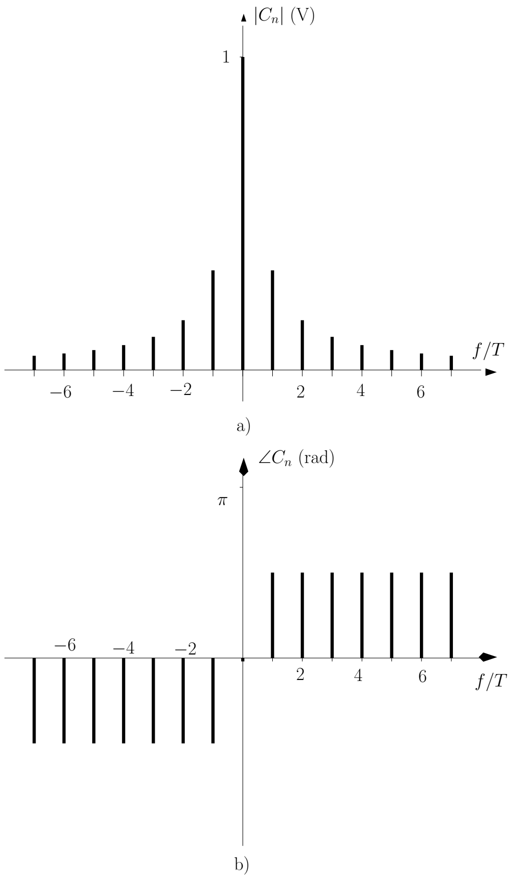


Figure 3.7: *Spectrum of* $v_2(t)$ *a) Magnitude. b) Phase.*

Solution of problem 3.6

Using phasor analysis we determine the output voltage V_R as follows:

$$V_R = V_S \frac{R}{Z_C + R} \tag{3.10}$$

where Z_C is the impedance associated with the capacitor:

$$Z_C = \frac{1}{j\,2\,\pi\,f\,C}$$

Eqn 3.10 can be written as:

$$V_R = V_S \frac{j\,2\,\pi\,f\,R\,C}{1 + j\,2\,\pi\,f\,R\,C} \tag{3.11}$$

we define

$$H(f) = \frac{V_R}{V_S}$$

giving

$$= \frac{j\,2\,\pi\,f\,R\,C}{1 + j\,2\,\pi\,f\,R\,C} \tag{3.12}$$

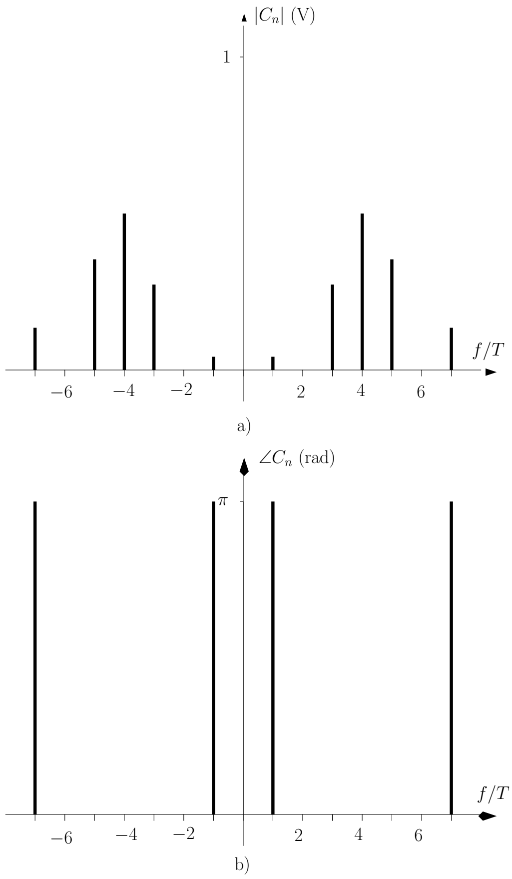


Figure 3.8: *Spectrum of* $v_3(t)$ *a) Magnitude. b) Phase.*

Solution of problem 3.7

- a): Using the result of the last problem (see eqn 3.11 with $V_O = V_R$) and by applying the Superposition theorem, that is, substituting the phasor V_S in eqn 3.11 by the phasors C_n given by eqn 3.8 and 3.9, we obtain the phasors representing the output voltage, V_{O_n}, as follows:

$$V_{O_n} = \begin{cases} 0 & , \quad n = 0 \\ \dfrac{-V_B\, R\, C}{T\left(1 + j\, 2\, \pi\, \dfrac{n}{T}\, R\, C\right)} & , \quad n \neq 0 \end{cases} \tag{3.13}$$

  and the output voltage, $v_o(t)$ can be written as :

$$\begin{aligned} v_o(t) &= \sum_{n=1}^{\infty} |2\, V_{O_n}| \cos\left(2\, \pi\, \frac{n}{T}\, t + \angle V_{O_n}\right) \\ &= \sum_{n=1}^{\infty} \frac{5.6}{\sqrt{1 + (2.8\, \pi\, n)^2}} \cos\left[2\, \pi\, \frac{n}{T}\, t + \pi - \tan^{-1}(2.8\, \pi\, n)\right] \text{ V} \end{aligned} \tag{3.14}$$

- b): Using phasor analysis we determine the output voltage V_O as follows:

$$V_O = V_S\, \frac{Z_L}{Z_L + R} \tag{3.15}$$

  where Z_L is the impedance associated with the inductance:

$$Z_L \;=\; j\, 2\, \pi\, f\, L$$

Eqn 3.15 can be written as:

$$V_O = V_S \, \frac{j\,2\,\pi\,f\,L/R}{1 + j\,2\,\pi\,f\,L/R} \tag{3.16}$$

Using again the Superposition theorem, that is, substituting the phasor V_S in eqn 3.16 by the phasors C_n given by eqn 3.8 and 3.9, we obtain the phasors V_{O_n} as follows:

$$V_{O_n} = \begin{cases} 0 & , \quad n = 0 \\ \dfrac{-V_B\,L/R}{T\,\left(1 + j\,2\,\pi\,\dfrac{n}{T}\,L/R\right)} & , \quad n \neq 0 \end{cases} \tag{3.17}$$

and the output voltage, $v_o(t)$ can be written as :

$$\begin{aligned} v_o(t) &= \sum_{n=1}^{\infty} |2\,V_{O_n}|\,\cos\left(2\,\pi\,\frac{n}{T}\,t + \angle V_{O_n}\right) \\ &= \sum_{n=1}^{\infty} \frac{5.6}{\sqrt{1 + (2.8\,\pi\,n)^2}}\,\cos\left[2\,\pi\,\frac{n}{T}\,t + \pi - \tan^{-1}(2.8\,\pi\,n)\right] \ \text{V} \end{aligned} \tag{3.18}$$

Comparing eqn 3.14 with eqn 3.18 we observe that these are identical. This means that both circuits produce the same output when driven by the same input signal, regardless of frequency. Both circuits are high-pass filters.

Solution of problem 3.8

We apply phasor analysis together with the Nodal analysis method to the circuits of figure 3.9.

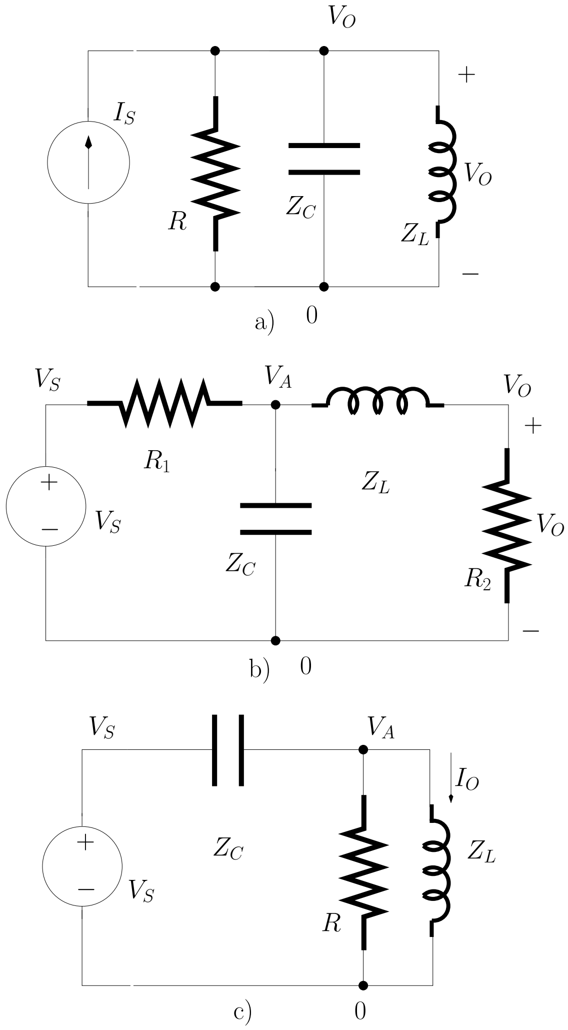


Figure 3.9: *Circuits of problem 3.8.*

- *Circuit a):* For this circuit we observe that the output voltage is applied to the parallel combination of Z_C, Z_L and R. This parallel combination can be represented by an equivalent impedance Z_{eq} given by:

$$\begin{aligned} Z_{eq} &= Z_C||Z_L||R \\ &= \frac{j\,2\,\pi\,f\,L\,R}{R(1-(2\,\pi\,f)^2\,L\,C+j\,2\,\pi\,fL} \end{aligned} \tag{3.19}$$

Since $V_O = I_S\,Z_{eq}$ the transfer function is equal to Z_{eq};

$$H(f) = \frac{j\,2\,\pi\,f\,L\,R}{R(1-(2\,\pi\,f)^2\,L\,C+j\,2\,\pi\,fL} \tag{3.20}$$

The magnitude of the transfer function is:

$$|H(f)| = \frac{2\,\pi\,f\,L\,R}{\sqrt{R^2[1-(2\,\pi\,f)^2LC]^2+(2\,\pi\,f)^2\,L^2}} \tag{3.21}$$

Figure 3.10 shows the magnitude of the transfer function versus the frequency.

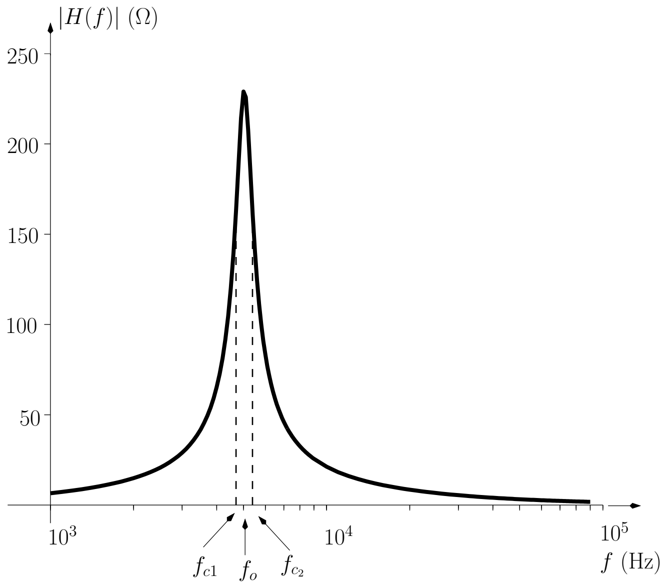


Figure 3.10: *Magnitude of the transfer function versus the frequency.*

Note that the peak value of the magnitude of the transfer function is equal to $R = 230\ \Omega$ and it occurs at the central frequency f_o given by:

$$\begin{aligned} f_o &= \frac{1}{2\,\pi\sqrt{LC}} \\ &= 5.03 \text{ kHz} \end{aligned}$$

The 3 dB cut-off frequencies satisfy the following eqn :

$$\frac{2\,\pi\,f\,L\,R}{\sqrt{R^2[1-(2\,\pi\,f)^2LC]^2+(2\,\pi\,f)^2\,L^2}} \quad = \quad \frac{R}{\sqrt{2}}$$

Solving, we obtain:

$$(2\pi f)^2 \quad = \quad \frac{1+2\,\tau_C/\tau_L}{2\,\tau_C^2} \pm \frac{\sqrt{1+4\,\tau_C/\tau_L}}{2\,\tau_C^2}$$

where $\tau_C = R\,C$ and $\tau_L = L/R$. Taking the square root we find:

$$\begin{aligned} f_{c_1} &= \frac{\left(1+2\,\tau_C/\tau_L-\sqrt{1+4\,\tau_C/\tau_L}\right)^{\frac{1}{2}}}{2\,\pi\,\sqrt{2}\,\tau_C} \\ f_{c_2} &= \frac{\left(1+2\,\tau_C/\tau_L+\sqrt{1+4\,\tau_C/\tau_L}\right)^{\frac{1}{2}}}{2\,\pi\,\sqrt{2}\,\tau_C} \end{aligned}$$

The bandwidth is $(f_{c_2} - f_{c_1}) = 692$ Hz. The Quality factor can be calculated as indicated below:

$$\begin{aligned} Q &= \frac{f_o}{f_{c_2}-f_{c_1}} \\ &= 7.3 \end{aligned}$$

- *Circuit b):* For this circuit we can write the following set of eqns:

$$\begin{cases} \dfrac{V_S-V_A}{R_1} = \dfrac{V_A}{Z_C} + \dfrac{V_A-V_O}{Z_L} \\ V_O = \dfrac{R_2}{R_2+Z_L}\,V_A \end{cases} \tag{3.22}$$

with

$$Z_C = \frac{1}{j\,\omega\,C}$$
$$Z_L = j\,\omega\,L$$

Solving we obtain:

$$V_O = V_S \frac{R_2}{R_2 + R_1 + j\,2\,\pi\,f\,(R_2\,R_1\,C + L) - L\,R_1\,C\,(2\,\pi\,f)^2}$$

that is:

$$H(f) = \frac{R_2}{R_2 + R_1 + j\,2\,\pi\,f\,(R_2\,R_1\,C + L) - L\,R_1\,C\,(2\,\pi\,f)^2}$$

Figure 3.11 shows the magnitude of the transfer function versus the frequency. The 3 dB

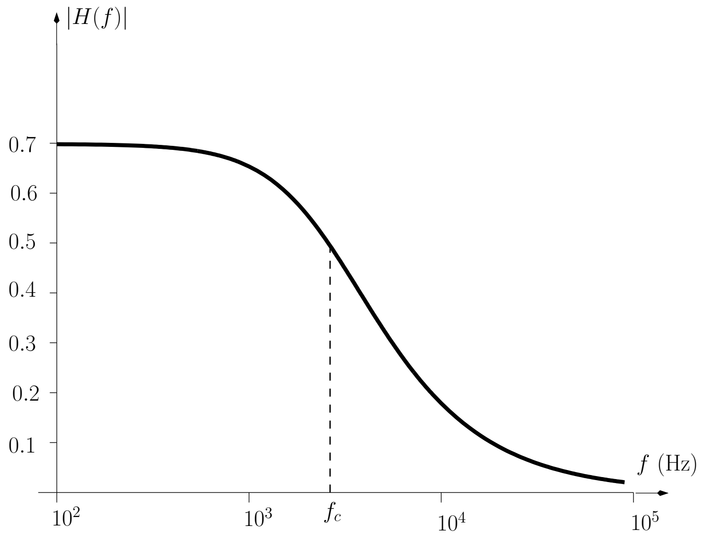


Figure 3.11: *Magnitude of the transfer function versus the frequency.*

cut-off frequency satisfies the following eqn :

$$|H(f)| = \frac{1}{\sqrt{2}} \frac{R_2}{R_2 + R_1}$$

From figure 3.11 we can obtain the 3 dB cut-off frequency; $f_c = 2.64$ kHz. The bandwidth is also 2.64 kHz.

- *Circuit c):* For this circuit we can write;

$$V_O = \frac{Z_{LR}}{Z_{LR} + Z_C} V_S$$

where Z_{LR} is the impedance associated with the parallel combination of R with L, that is:

$$Z_{LR} = \frac{j\,2\,\pi\,f\,L\,R}{j\,2\,\pi\,f\,L + R}$$

The output current I_O can be expressed as:

$$I_O = \frac{V_O}{Z_L}$$

that is:

$$I_O = V_S \frac{Z_{LR}}{Z_{LR} + Z_C} \frac{1}{Z_L}$$

Finally,

$$
\begin{aligned}
H(f) &= \frac{Z_{LR}}{Z_{LR}+Z_C}\,\frac{1}{Z_L} \\
&= \frac{j\,2\,\pi\,f\,C}{j\,2\,\pi\,f\,L + R\,[1-(2\,\pi\,f)^2 L\,C]}
\end{aligned}
$$

Figure 3.12 shows the magnitude of the transfer function versus the frequency. The 3 dB

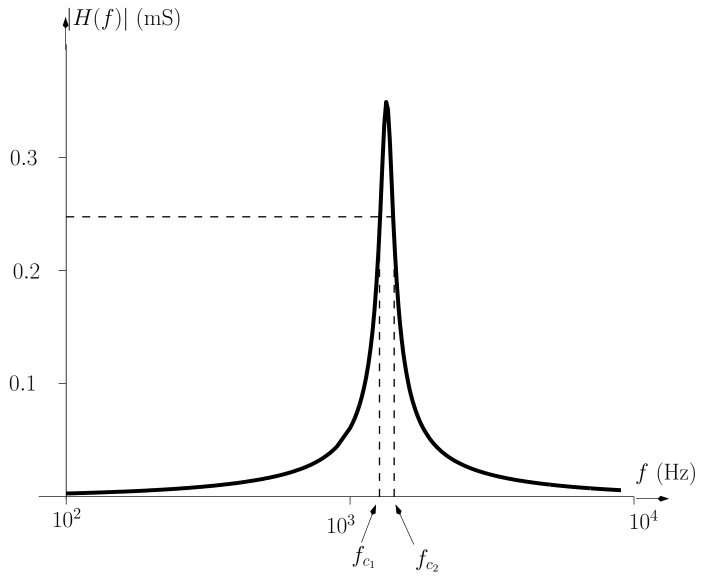


Figure 3.12: *Magnitude of the transfer function versus the frequency.*

cut-off frequency satisfy the following eqn :

$$
|H(f)| = \frac{1}{\sqrt{2}}\,\frac{C}{L}
$$

From figure 3.12 we can obtain the 3 dB cut-off frequencies; $f_{c_1} = 1.27$ kHz and $f_{c_2} = 1.43$ kHz. The bandwidth is therefore 160 Hz.

Solution of problem 3.9

- *Signal a):* $v_1(t)$ can be expressed as

$$v_1(t) = \begin{cases} \frac{b}{a}\,t & \text{for } 0 < t < a \\ 0 & \text{elsewhere} \end{cases} \tag{3.23}$$

with $b = 3/2$ and $a = 4 \times 10^{-3}$.
$V_1(f)$ can be calculated as follows:

$$\begin{aligned} V_1(f) &= \int_{-\infty}^{\infty} v_1(t)\, e^{-j\,2\,\pi\,f\,t}\, dt \\ &= \int_0^a \frac{b}{a}\, t\, e^{-j\,2\,\pi\,f\,t}\, dt \\ &= b\, \frac{(1 + 2\,j\,a\,\pi\,f)\, e^{-2\,j\,a\,\pi\,f} - 1}{4\,a\,\pi^2\,f^2} \end{aligned}$$

Figure 3.13 a) shows the magnitude of $V_1(f)$ and figure 3.13 b) shows the phase of $V_1(f)$.

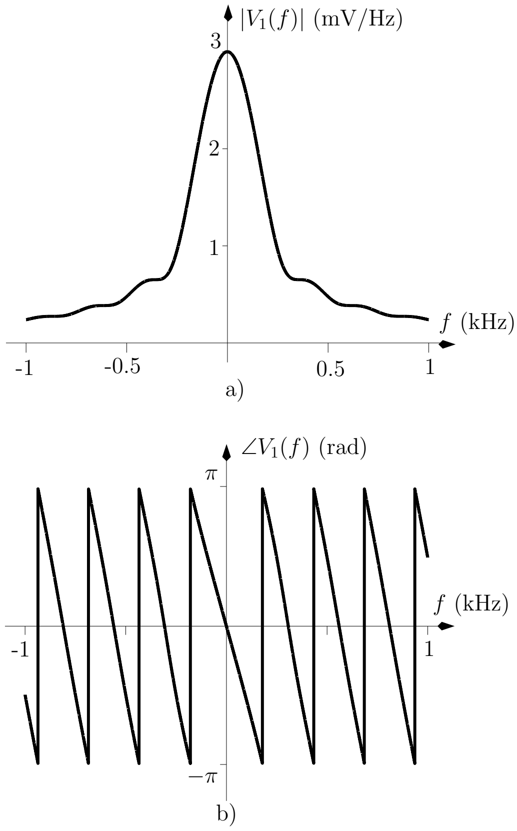


Figure 3.13: *a) Magnitude of* $V_1(f)$. *b) Phase of* $V_1(f)$.

- *Signal b):* $v_2(t)$ can be expressed as

$$v_2(t) = \begin{cases} (t + 3a)/a & \text{for } -3a < t < -2a \\ 1 & \text{for } -2a \le t \le 2a \\ (-t + 3a)/a & \text{for } 2a < t < 3a \\ 0 & \text{elsewhere} \end{cases} \tag{3.24}$$

with $a = 10^{-3}$.

$V_2(f)$ can be calculated as follows:

$$\begin{aligned}
V_2(f) &= \int_{-\infty}^{\infty} v_2(t)\, e^{-j\,2\,\pi\,f\,t}\, dt \\
&= \int_{-3a}^{-2a} \frac{t+3a}{a}\, e^{-j\,2\,\pi\,f\,t}\, dt + \int_{-2a}^{2a} e^{-j\,2\,\pi\,f\,t}\, dt \\
&+ \int_{2a}^{3a} \frac{-t+3a}{a}\, e^{-j\,2\,\pi\,f\,t}\, dt \\
&= \frac{2\,\cos(4\,\pi\,f\,a) - 2\,\cos(6\,\pi\,f\,a) - 4\,\pi\,f\,a\sin(4\,\pi\,f\,a)}{4\,\pi^2\,f^2\,a} \\
&+ 4\,a\ \mathrm{sinc}\,(4\,f\,a)
\end{aligned}$$

Figure 3.14 a) shows the magnitude of $V_2(f)$ and figure 3.14 b) shows the phase of $V_2(f)$.

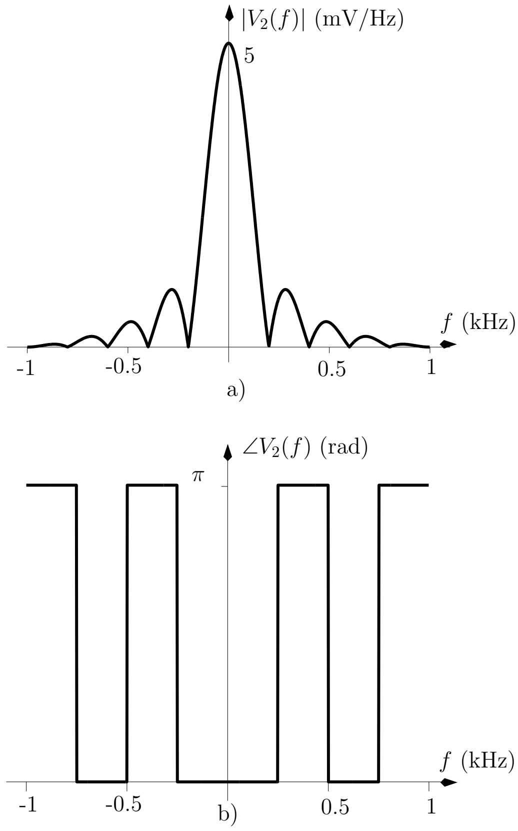


Figure 3.14: *a) Magnitude of* $V_2(f)$*. b) Phase of* $V_2(f)$*.*

- *Signal c):* $v_3(t)$ can be expressed as

$$v_3(t) = 2\,\mathrm{rect}\left(\frac{t}{2\,a}\right) + \mathrm{rect}\left(\frac{t-2\,a}{2\,a}\right) + \mathrm{rect}\left(\frac{t+2\,a}{2\,a}\right)$$

with $a = 10^{-3}$.

$V_3(f)$ can be calculated as follows:

$$\begin{aligned}
V_3(f) &= \int_{-\infty}^{\infty} v_3(t)\, e^{-j\,2\,\pi\,f\,t}\, dt \\
&= 4\,a\,\mathrm{sinc}\,(2\,f\,a)\ [1+\cos(4\,\pi\,f\,a)]
\end{aligned}$$

Figure 3.15 a) shows the magnitude of $V_3(f)$ and figure 3.15 b) shows the phase of $V_3(f)$.

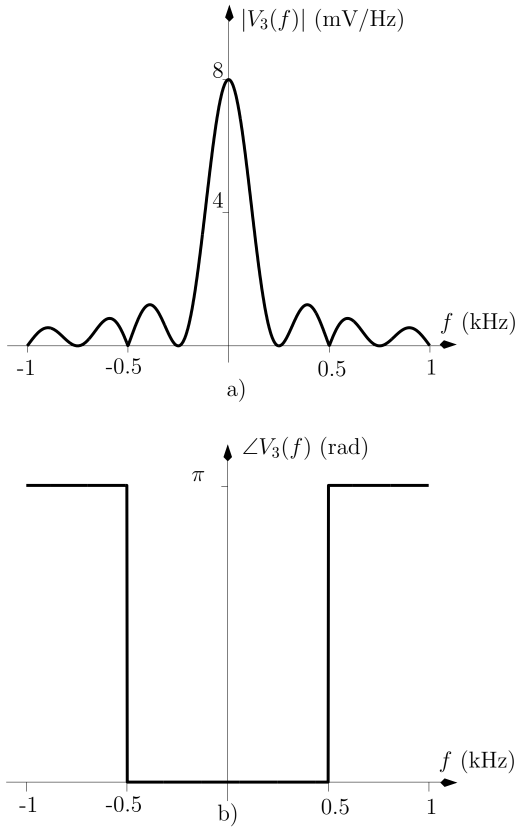


Figure 3.15: *a) Magnitude of* $V_3(f)$*. b) Phase of* $V_3(f)$*.*

Solution of problem 3.10

Let us consider the convolution between two functions $x(t)$ and $y(t)$. $x(t) * y(t)$ can be expressed as:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, y(\lambda)\, d\lambda \tag{3.25}$$

Using the variable transformation

$$\tau = t - \lambda$$

we have

$$\begin{aligned} d\tau &= -d\lambda \\ \lambda \to -\infty &\Rightarrow \tau \to \infty \\ \lambda \to \infty &\Rightarrow \tau \to -\infty \end{aligned}$$

and eqn 3.25 can be written as follows:

$$\begin{aligned} x(t) * y(t) &= -\int_{+\infty}^{-\infty} x(\tau)\, y(t-\tau)\, d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau \\ &= y(t) * x(t) \end{aligned}$$

Solution of problem 3.11

The transfer function of the circuit can be calculated using phasor analysis:

$$\begin{aligned} H(f) &= \frac{V_O}{V_S} \\ &= \frac{1}{1 + j\,2\,\pi\,f\,R\,C - (2\,\pi\,f)^2\,L\,C} \end{aligned}$$

This transfer function can be written as follows:

$$H(f) = \frac{\omega_o^2}{\omega_o^2 + j\,2\,\pi\,f\,(2\,\eta\,\omega_o) - (2\,\pi\,f)^2}$$

where

$$\begin{aligned} \omega_o &= \frac{1}{\sqrt{L\,C}} \\ &= 30.9 \text{ krad/s} \end{aligned}$$

and

$$\begin{aligned} \eta &= \frac{1}{2}\,R\,\sqrt{\frac{C}{L}} \\ &= 0.7 \end{aligned}$$

The transfer function can also be written as

$$H(f) = \frac{\beta_2\,\beta_1}{\beta_2 - \beta_1}\left(\frac{1}{\beta_1 + j\,2\,\pi\,f} - \frac{1}{\beta_2 + j\,2\,\pi\,f}\right)$$

with

$$\begin{aligned} \beta_1 &= \eta\,\omega_o \pm \omega_o\,\sqrt{\eta^2 - 1} \\ \beta_2 &= \eta\,\omega_o \mp \omega_o\,\sqrt{\eta^2 - 1} \end{aligned}$$

Choosing $\beta_1 = \eta\,\omega_o - \omega_o\,\sqrt{\eta^2 - 1}$ and $\beta_2 = \eta\,\omega_o + \omega_o\,\sqrt{\eta^2 - 1}$ and using the table of Fourier transforms in Appendix A we can write the impulse response as:

$$\begin{aligned} h(t) &= \frac{\beta_2\,\beta_1}{\beta_2 - \beta_1}\left(e^{-\beta_1\,t} - e^{-\beta_2\,t}\right)\,u(t) \\ &= \frac{\omega_o}{2\,\sqrt{\eta^2 - 1}} e^{-\omega_o\,\eta\,t}\left(e^{\omega_o\,\sqrt{\eta^2-1}\,t} - e^{-\omega_o\,\sqrt{\eta^2-1}\,t}\right)\,u(t) \end{aligned}$$

Since η is less than one, the last eqn can be written as:

$$\begin{aligned} h(t) &= \frac{\omega_o}{j\,2\,\sqrt{1 - \eta^2}} e^{-\omega_o\,\eta\,t}\left(e^{j\omega_o\,\sqrt{1-\eta^2}\,t} - e^{-j\omega_o\,\sqrt{1-\eta^2}\,t}\right)\,u(t) \\ &= \frac{\omega_o}{\sqrt{1 - \eta^2}} e^{-\omega_o\,\eta\,t}\,\sin\left(\omega_o\,\sqrt{1 - \eta^2}\,t\right)\,u(t) \end{aligned}$$

Figure 3.16 shows $h(t)$.

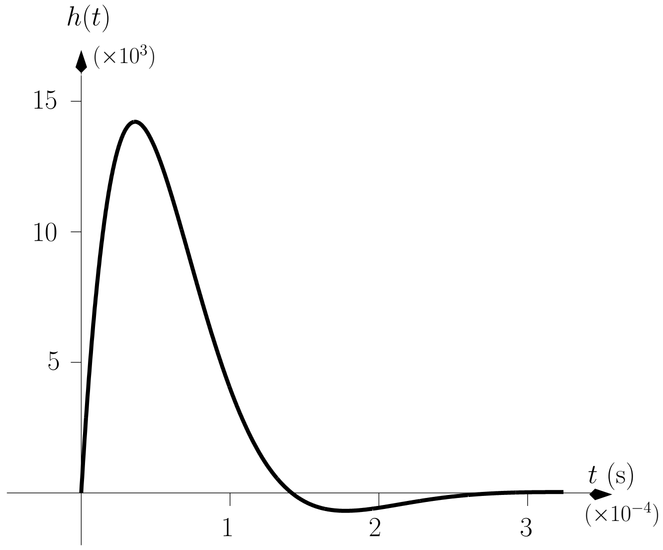


Figure 3.16: *Impulse response.*

Solution of problem 3.12

The output voltage is given by:

$$\begin{aligned} v_o(t) &= h(t) * v_i(t) \\ &= \int_{-\infty}^{\infty} h(\lambda)\, v_i(t-\lambda) d\lambda \end{aligned} \tag{3.26}$$

where $h(t)$ is as calculated in the previous exercise and $v_i(t)$ is given by

$$v_i = V_a \text{rect}\left(\frac{t - T_a/2}{T_a}\right)$$

Eqn 3.26 can be written as:

$$v_o(t) = \begin{cases} \displaystyle\int_0^t h(\lambda)\, V_a\, d\lambda & \text{for} \quad < t < T_a \\ \displaystyle\int_{t-T_a}^t h(\lambda)\, V_a\, d\lambda & \text{for} \quad t \geq T_a \end{cases} \tag{3.27}$$

Solving, we obtain:

$$v_o(t) = \begin{cases} V_a - \dfrac{V_a}{\sqrt{1-\eta^2}} e^{-\omega_o \eta t} \sin\left(\sqrt{1-\eta^2}\, \omega_o t + \phi\right), & 0 < t < T_a \\ \dfrac{V_a}{\sqrt{1-\eta^2}} \Big[e^{-\omega_o \eta t} \sin\left(\sqrt{1-\eta^2}\, \omega_o t + \phi\right) & \\ \quad - e^{-\omega_o \eta (t-T_a)} \sin\left(\sqrt{1-\eta^2}\, \omega_o (t-T_a) + \phi\right)\Big], & t \geq T_a \end{cases} \tag{3.28}$$

with

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\eta^2}}{\eta}\right)$$

Figure 3.17 shows $v_i(t)$ and $v_o(t)$.

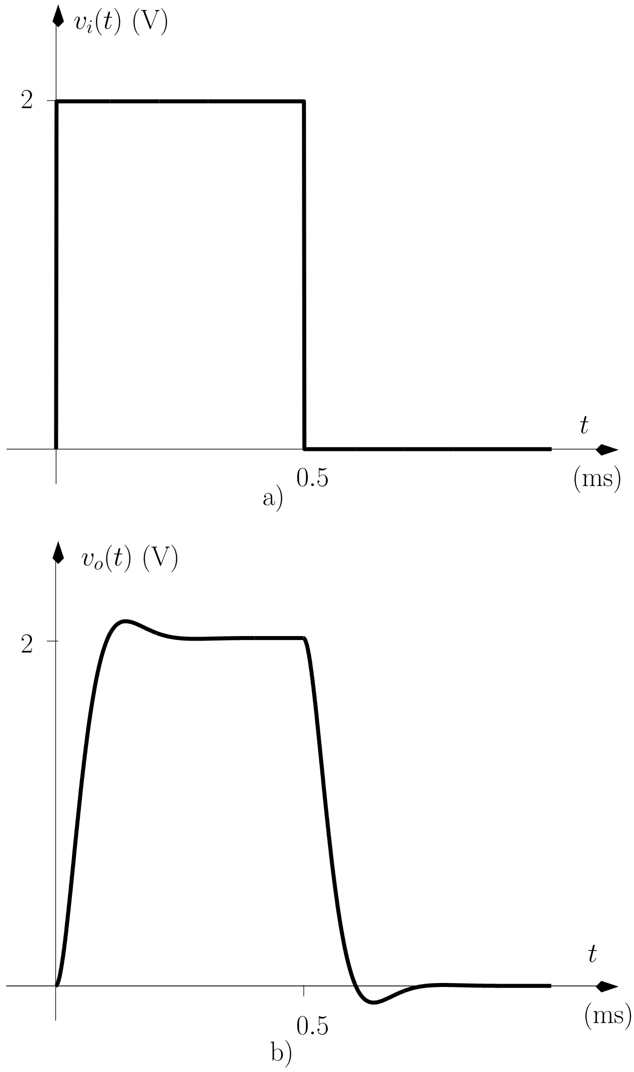


Figure 3.17: *a)* $v_i(t)$. *b)* $v_o(t)$.

Chapter 4

Natural and forced responses circuit analysis

Solution of problem 4.1

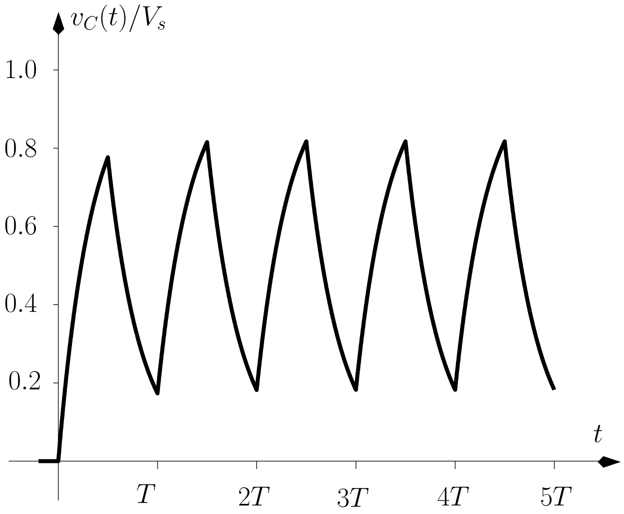


Figure 4.1: *Voltage normalised to V_s versus time normalised to T.*

Figure 4.1 shows $v_C(t)$ when $\tau = T/3$. From this figure we observe that the circuit reaches its steady-state at about $2T$. We conclude that for a smaller time constant the circuit bandwidth increases and, therefore, the circuit reaches its steady-state more quickly.

Solution of problem 4.2

- The Laplace transform of $v_1(t)$ can be calculated as follows:

$$\begin{aligned} V_1(s) &= \int_0^\infty v_1(t)\, e^{-s\,t}\, dt \\ &= \int_0^1 t\, e^{-s\,t}\, dt + \int_0^\infty e^{-s\,t}\, dt \\ &= \left[-\frac{t}{s}\, e^{-s\,t} - \frac{1}{s^2}\, e^{-s\,t}\right]_0^1 + \left[-\frac{1}{s}\, e^{-s\,t}\right]_0^\infty \\ &= \frac{1}{s^2}\left(1 - e^{-s}\right) \quad \text{Real}(s) > 0 \end{aligned}$$

- The Laplace transform of $v_2(t)$ can be calculated as follows:

$$\begin{aligned} V_2(s) &= \int_0^\infty v_2(t)\, e^{-s\,t}\, dt \\ &= \int_0^\infty t^2\, e^{-s\,t}\, dt \\ &= \left[\frac{t^2}{s}\, e^{-s\,t} - \frac{2\,t}{s^2}\, e^{-s\,t} - \frac{2}{s^3}\, e^{-s\,t}\right]_0^\infty \\ &= \frac{2}{s^3} \quad \text{Real}(s) > 0 \end{aligned}$$

- $v_3(t)$ can be expressed as follows:

$$v_3(t) = t^2\, u(t) - (t-1)^2\, u(t-1)$$

  Now the Laplace transform of $v_2(t)$ can be calculated as follows:

$$\begin{aligned} V_1(s) &= V_2(s) - V_2(s)\, e^{-s} \\ &= \frac{2}{s^3}\left(1 - e^{-s}\right) \quad \text{Real}(s) > 0 \end{aligned}$$

Solution of problem 4.3

- The roots of the equation $(s-a)^2+b^2=0$ are:

$$s = a \pm j\,b$$

Now we can write:

$$\frac{1}{(s-a)^2+b^2} = \frac{K_1}{s-a-j\,b}+\frac{K_2}{s-a+j\,b}$$

that is:

$$\frac{1}{(s-a)^2+b^2} = \frac{K_1\,(s-a+j\,b)+K_2(s-a-j\,b)}{(s-a)^2+b^2}$$

The last eqn is an equality if:

$$\begin{cases} K_1+K_2=0 \\ K_1\,(-a+j\,b)+K_2\,(-a-j\,b)=1 \end{cases} \tag{4.1}$$

Solving, we obtain:

$$\begin{aligned} K_1 &= \frac{1}{j\,2\,b} \\ K_2 &= -\frac{1}{j\,2\,b} \end{aligned}$$

The inverse Laplace transform can now be obtained as follows:

$$\begin{aligned} x_1(t) &= \frac{1}{j\,2\,b}\,e^{(a+j\,b)\,t}-\frac{1}{j\,2\,b}\,e^{(a-j\,b)\,t} \\ &= \frac{1}{b}\,e^{a\,t}\,\frac{e^{j\,b\,t}-e^{-j\,b\,t}}{2\,j} \\ &= \frac{1}{b}\,e^{a\,t}\,\sin(b\,t) \end{aligned}$$

- We can write $X_2(s)$ as follows:

$$\frac{s}{(s+a)^2\,(s+b)^2} = \frac{K_{a_1}}{(s+a)^2}+\frac{K_{a_2}}{s+a}+\frac{K_{b_1}}{(s+b)^2}+\frac{K_{b_2}}{s+b}$$

The coefficients K_{a_1}, K_{a_2}, K_{b_1} and K_{b_2} can be obtained solving the following set of eqns:

$$\begin{cases} K_{a_2}+K_{b_2}=0 \\ K_{a_2}\,a+2\,K_{a_2}\,b+K_{a_1}+K_{b_1}+K_{b_2}\,b+2\,K_{b_2}\,a=0 \\ 2\,K_{a_1}\,b+2\,K_{a_2}\,a\,b+K_{a_2}\,b^2+2\,K_{b_1}\,a+2\,K_{b_2}\,b\,a+K_{b_2}\,a^2=1 \\ K_{a_1}\,b^2+K_{b_2}\,b\,a^2+K_{a_2}\,a\,b^2+K_{b_1}\,a^2=0 \end{cases} \tag{4.2}$$

Solving, we obtain:

$$\begin{aligned} K_{a_1} &= \frac{-a}{(a-b)^2} \\ K_{b_1} &= \frac{-b}{(a-b)^2} \\ K_{a_2} &= \frac{-(a+b)}{(a-b)^3} \\ K_{a_3} &= \frac{(a+b)}{(a-b)^3} \end{aligned}$$

The inverse Laplace transform can be written as

$$x_2(t) = (K_{a_1}+K_{a_2}\,t)\,e^{-a\,t}\,u(t)+(K_{b_1}+K_{b_2}\,t)\,e^{-b\,t}\,u(t)$$

- $X_3(s)$ can be written as

$$\frac{1}{s^2 - a^2} = \frac{K_1}{s+a} + \frac{K_2}{s-a}$$

with

$$\begin{aligned} K_1 &= -\frac{1}{2\,a} \\ K_2 &= \frac{1}{2\,a} \end{aligned}$$

and $x_1(t)$ can be obtained as follows:

$$\begin{aligned} x_1(t) &= -\frac{1}{2\,a}\, e^{-a\,t}\, u(t) + \frac{1}{2\,a}\, e^{a\,t}\, u(t) \\ &= \frac{1}{a}\, \sinh(a\,t)\, u(t) \end{aligned}$$

Solution of problem 4.4

- *Circuit a):* Figure 4.2 a) shows the equivalent circuit for $t < 0$. Note that, if connected to a DC

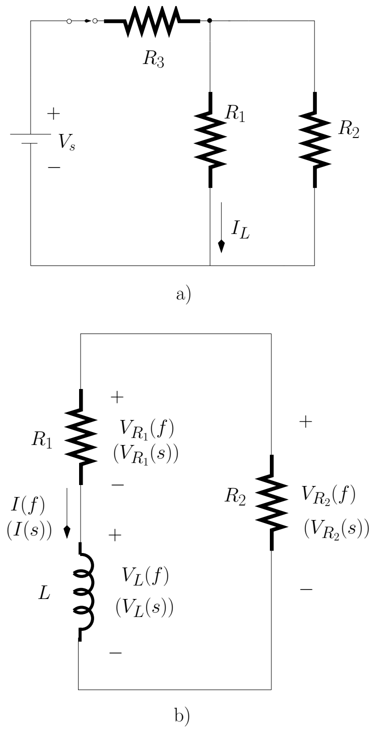


Figure 4.2: *a) Equivalent circuit for $t < 0$. b) Equivalent circuit for $t \geq 0$.*

source for a long time the inductor behaves as a short-circuit. The current that flows through R_1 and L can be determined as follows:

$$\begin{aligned} I_L &= \frac{V_s}{(R_1||R_2) + R_3} \frac{R_2}{R_1 + R_2} \\ &= 0.75 \text{ mA} \end{aligned} \tag{4.3}$$

The voltage across R_1 is:

$$\begin{aligned} V_{R_1} &= \frac{R_1}{R_1 + R_2} V_s \\ &= 0.75 \text{ V} \end{aligned}$$

Figure 4.2 b) shows the equivalent circuit for $t \geq 0$. Using Fourier transforms we can write the following eqn:

$$V_{R_1}(f) + V_L(f) = V_{R_2}(f)$$

that is:

$$R_1\, I(f) + j\, 2\, \pi\, f\, L\, I(f) - L\, I_L = -R_2\, I(f)$$

where I_L is given by eqn 4.3. Solving this eqn to obtain $I(f)$ we have:

$$I(f) = \frac{L\, I_L}{R_1 + R_2 + j\, 2\, \pi\, f\, L}$$

and $V_{R_1}(f)$ is:

$$\begin{aligned} V_{R_1}(f) &= R_1\, I(f) \\ &= \frac{R_1\, L\, I_L}{R_1 + R_2 + j\, 2\, \pi\, f\, L} \end{aligned}$$

Calculating the inverse Fourier transform we obtain the time domain voltage across R_1 for $t \geq 0$, that is:

$$v_{R_1}(t) \quad = \quad R_1\, I_L\, e^{-t\,(R_1+R_2)/L}\, u(t) \tag{4.4}$$

The voltage $v_{R_1}(t)$ can be written, for all time t, as follows:

$$v_{R_1}(t) \quad = \quad \frac{R_1}{R_1+R_2}\, V_s\, u(-t) + R_1\, I_L\, e^{-t\,(R_1+R_2)/L}\, u(t) \tag{4.5}$$

Now, we use Laplace transforms to analyse the circuit (see figure 4.2 b)). We can write the following eqn:

$$R_1\, I(s) + s\, L\, I(f) - L\, I_L = -R_2\, I(s)$$

that is:

$$I(f) \quad = \quad \frac{L\, I_L}{R_1 + R_2 + s\, L}$$

and $V_{R_1}(s)$ is:

$$\begin{aligned} V_{R_1}(f) \quad &= \quad R_1\, I(s) \\ &= \quad \frac{R_1\, L\, I_L}{R_1 + R_2 + s\, L} \end{aligned}$$

Calculating the inverse Laplace transform we obtain the time domain voltage across R_1 for $t \geq 0$, that is:

$$v_{R_1}(t) \quad = \quad R_1\, I_L\, e^{-t\,(R_1+R_2)/L}\, u(t) \tag{4.6}$$

Comparing eqn 4.4 with 4.6 we observe that these are equal. This is expected since both methods (Fourier and Laplace transforms) can be used to study the natural response of a circuit. Figure 4.3 shows the voltage across R_1.

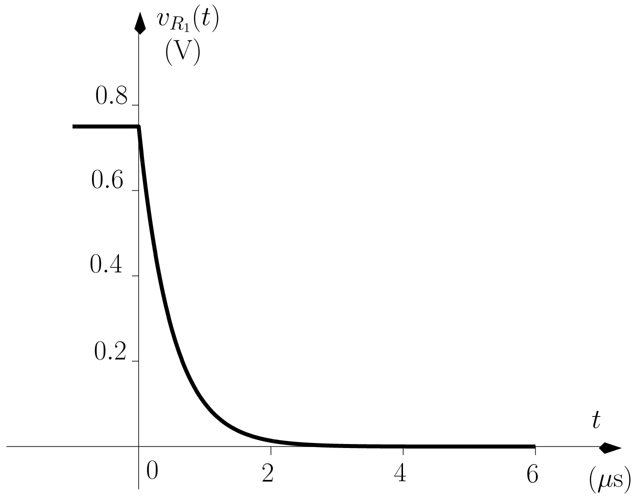


Figure 4.3: *Voltage across* R_1.

- *Circuit b):* Figure 4.4 a) shows the equivalent circuit for $t < 0$. Note that, for DC, the capacitor behaves as an open-circuit. Hence all the current flows through R_2 giving rise to a voltage across its terminals:

$$\begin{aligned} V_{R_2} \quad &= \quad R_2\, I_s \\ &= \quad 4\ \text{V} \end{aligned}$$

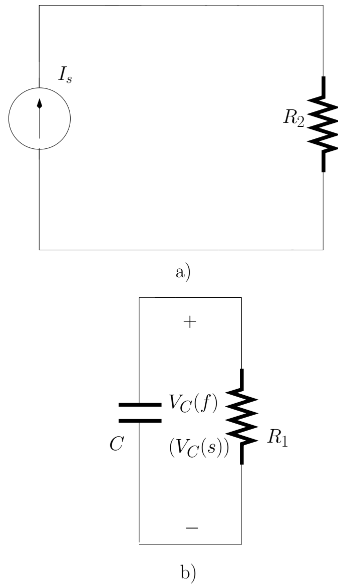


Figure 4.4: *a) Equivalent circuit for $t < 0$ b)Equivalent circuit for $t \geq 0$.*

Since the resistor R_1 is not conducting, $v_{R_1}(t) = 0$ for $t < 0$ and the voltage across the capacitor, for $t < 0$, is V_{R_2}.

Figure 4.4 b) shows the equivalent circuit for $t \geq 0$. From this circuit it is clear that the voltage across R_1 is the same as the voltage across the capacitor. Using Fourier transforms we can write:

$$\frac{V_C(f)}{R_1} = -j\,2\,\pi\,f\,C\,V_C(f) + C\,V_{R_2}$$

Note that V_{R_2} corresponds to the initial condition of the capacitor. The voltage $V_C(f)$ can be obtained as:

$$V_C(f) = V_{R_2}\,\frac{R_1\,C}{1 + j\,2\,\pi\,f\,R_1\,C}$$

Taking the inverse Fourier transform we get:

$$v_c(t) = V_{R_2}\,e^{-t/\tau}\,u(t)$$

with $\tau = R_1\,C$. The voltage across the resistance R_1, for all time t, can be written as:

$$v_{R_1}(t) = V_{R_2}\,e^{-t/\tau}\,u(t)$$

Using Laplace transforms we can write (for $t \geq 0$):

$$\frac{V_C(s)}{R_1} = -s\,C\,V_C(s) + C\,V_{R_2}$$

The voltage $V_C(s)$ can be obtained as:

$$V_C(s) = V_{R_2}\,\frac{R_1\,C}{1 + s\,R_1\,C}$$

Taking the inverse Laplace transform we get:

$$v_c(t) = V_{R_2}\,e^{-t/\tau}\,u(t)$$

where $\tau = R_1\,C$. The voltage across the resistance R_1 can be written, for all time t, as:

$$v_{R_1}(t) = V_{R_2}\,e^{-t/\tau}\,u(t)$$

Once again the Fourier transform and Laplace transforms give the same result. Figure 4.5 shows the voltage across R_1.

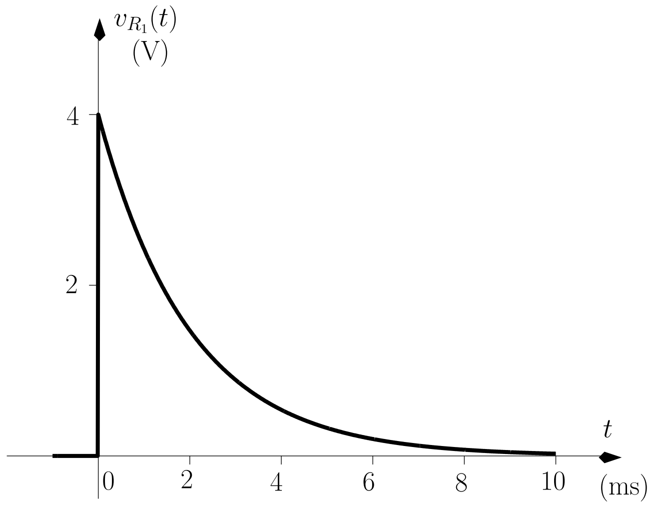


Figure 4.5: *Voltage across R_1.*

Solution of problem 4.5

- *Circuit a):* Figure 4.6 a) shows the equivalent circuit for $t < 0$. Since the capacitor represents

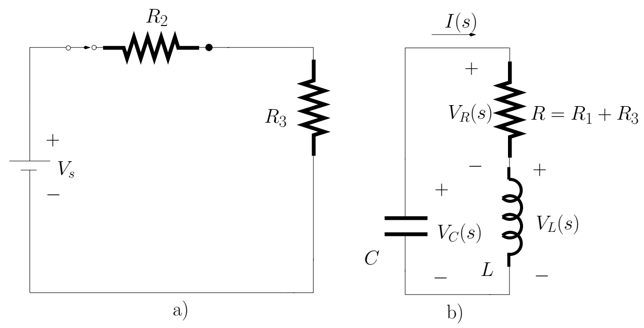


Figure 4.6: *a) Equivalent circuit for $t < 0$. b) Equivalent circuit for $t \geq 0$.*

an open-circuit at DC, R_1 is not conducting and the voltage across the capacitor, V_{co}, is the voltage across R_3, which is given by:

$$\begin{aligned} V_{R_3} &= V_s \frac{R_3}{R_2 + R_3} \\ &= 0.44 \text{ V} \end{aligned}$$

The inductor behaves as a short-circuit at DC. Therefore, the voltage across its terminals is zero. The current that flows through the inductor is the current that flows through R_3:

$$\begin{aligned} I_{lo} &= I_{R_3} \\ I_{R_3} &= \frac{V_{R_3}}{R_3} \\ &= 3.7 \text{ mA} \end{aligned}$$

Figure 4.6 b) shows the equivalent circuit for $t \geq 0$. For this circuit we can write (using Laplace domain analysis):

$$V_C(s) = V_R(s) + V_L(s)$$

that is

$$-\frac{I(s)}{s\,C} + \frac{V_{co}}{s} = R\,I(s) + s\,L\,I(s) - L\,I_{lo}$$

with $R = R_1 + R_3$. Solving in order to obtain $I(s)$ we get:

$$I(s) \quad = \quad = \frac{C\,V_{co} + L\,C\,s\,I_{lo}}{1 + s\,C\,R + L\,C\,s^2}$$

The last eqn can be written as follows:

$$\begin{aligned} I(s) \quad &= \quad C\,V_{co}\,\frac{\omega_n^2}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \\ &+ \quad I_{lo}\,\frac{s}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \end{aligned}$$

with

$$\begin{aligned} \omega_n \quad &= \quad \frac{1}{\sqrt{L\,C}} \\ &= \quad 50 \;\text{ krad/s} \\ \eta \quad &= \quad \frac{1}{2}\,R\,\sqrt{\frac{C}{L}} \\ &= \quad 2.2 \end{aligned}$$

Taking the inverse Laplace Transform ($\eta > 1$) we have:

$$\begin{aligned} i(t) \quad &= \quad \underbrace{C\,V_{co}\,\frac{\omega_n}{\sqrt{\eta^2 - 1}}\,\sinh\left(\omega_n\,\sqrt{\eta^2 - 1}\,t\right)\,e^{-t\,\eta\,\omega_n}\,u(t)}_{\text{Contribution from } V_{co}} \\ &+ \quad \underbrace{I_{lo}\left[\cosh\left(\omega_n\,\sqrt{\eta^2 - 1}\,t\right) - \frac{\eta}{\sqrt{\eta^2 - 1}}\,\sinh\left(\omega_n\,\sqrt{\eta^2 - 1}\,t\right)\right]\,e^{-t\,\eta\,\omega_n}\,u(t)}_{\text{Contribution from } I_{lo}} \end{aligned}$$

Figure 4.7 shows $i(t)$ versus the time.

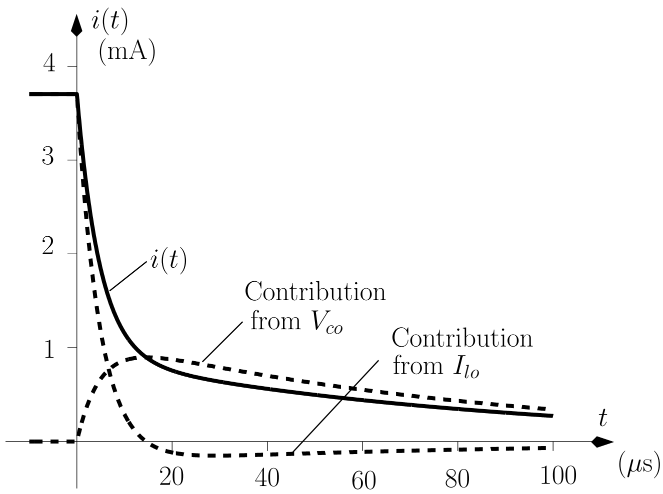


Figure 4.7: *The current for all time t.*

- *Circuit b):* Figure 4.8 a) shows the equivalent circuit for $t < 0$. The voltage across the capacitor is equal to the voltage across R_1 given by:

$$\begin{aligned} V_{co} \quad &= \quad V_{R_1} \\ &= \quad I_s\,R_1 \end{aligned}$$

The current through the inductor is zero since the capacitor is not conducting.

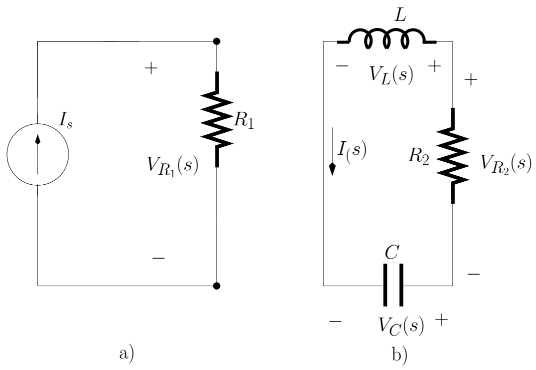


Figure 4.8: *a) Equivalent circuit for $t < 0$. b) Equivalent circuit for $t \geq 0$.*

Figure 4.8 b) shows the equivalent circuit for $t \geq 0$. For this circuit we can write:

$$V_L(s) = V_{R_2} + V_C(s)$$

that is

$$s\,L\,I(s) = -R_2\,I(s) - \frac{I(s)}{s\,C} + \frac{V_{co}}{s}$$

Solving, we get:

$$\begin{aligned} I(s) &= \frac{C\,V_{co}}{s^2\,L\,C + s\,C\,R_2 + 1} \\ &= C\,V_{co}\,\frac{\omega_n^2}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \end{aligned}$$

with

$$\begin{aligned} V_{co} &= I_s\,R_1 \\ &= 2 \;\text{ V} \\ \omega_n &= \frac{1}{\sqrt{L\,C}} \\ &= 14.1 \;\text{ krad/s} \\ \eta &= \frac{1}{2}\,R_2\,\sqrt{\frac{C}{L}} \\ &= 0.35 \end{aligned}$$

Taking the inverse Laplace transform we obtain

$$i(t) = C\,V_{co}\,\frac{\omega_n}{\sqrt{1-\eta^2}}\,\sin\left(\omega_n\,\sqrt{1-\eta^2}\,t\right)\,e^{-t\,\eta\,\omega_n}\,u(t)$$

Figure 4.9 shows $i(t)$.

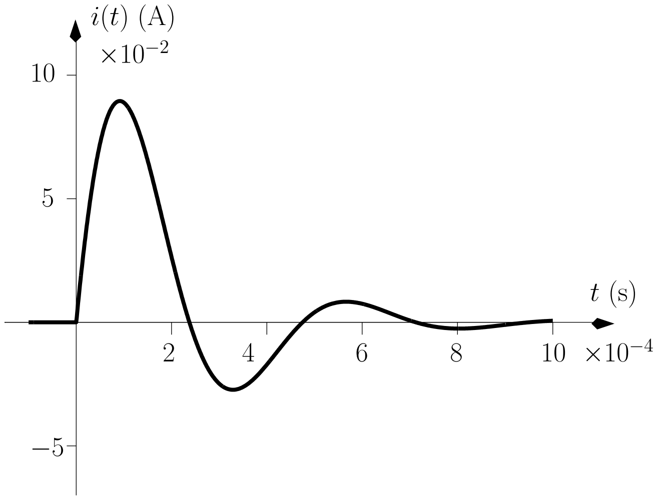


Figure 4.9: *The current $i(t)$.*

Solution of problem 4.6

- *Circuit a):* Figure 4.10 shows the equivalent circuit in the s-domain. For this circuit we can

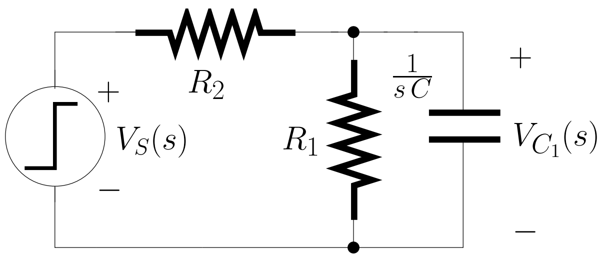


Figure 4.10: *Equivalent circuit in the s-domain.*

write:

$$\begin{aligned} V_{C_1}(s) &= V_S(s)\,\frac{R_1||\frac{1}{s\,C_1}}{\left(R_1||\frac{1}{s\,C_1}\right)+R_2} \\ &= V_S(s)\,\frac{R_1}{R_1+R_2}\times\frac{1}{1+s\,\tau} \end{aligned}$$

with

$$V_S(s) = \frac{V_s}{s}$$
$$\tau = C_1\,\frac{R_1\,R_2}{R_1+R_2}$$

$V_s = 3$ V. Thus $V_{C_1}(s)$ can be written as:

$$V_{C_1}(s) = \frac{V_s}{s}\times\frac{R_1}{R_1+R_2}\times\frac{1}{1+s\,\tau}$$

Taking the inverse Laplace transform we obtain

$$v_{C_1}(t) = \frac{V_s\,R_1}{R_1+R_2}\left(1-e^{-t/\tau}\right)u(t)$$

Figure 4.11 shows $v_{C_1}(t)$

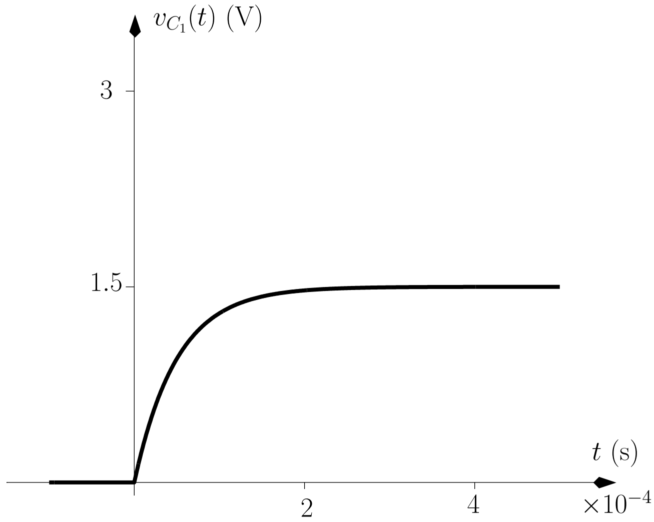


Figure 4.11: *The voltage across the capacitor C_1.*

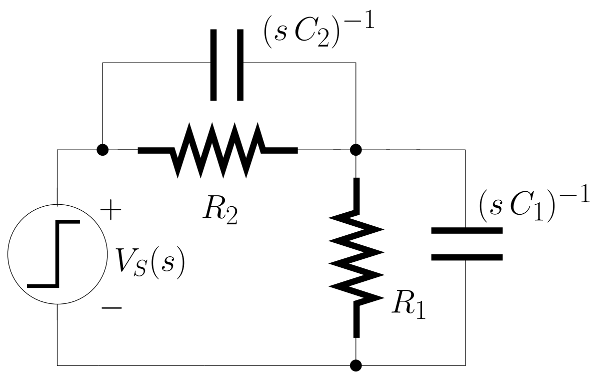


Figure 4.12: *Equivalent circuit in the s-domain.*

- *Circuit b):* Figure 4.12 shows the equivalent circuit in the s-domain. For this circuit we can write:

$$V_{C_1}(s) = V_S(s)\,\frac{R_1||\frac{1}{s\,C_1}}{\left(R_1||\frac{1}{s\,C_1}\right)+\left(R_2||\frac{1}{s\,C_2}\right)}$$

that is:

$$V_{C_1}(s) = V_S(s)\,\frac{R_1}{R_1+R_2}\,\frac{1+s\,R_2\,C_2}{1+s\,(C_1+C_2)\frac{R_1\,R_2}{R_1+R_2}}$$

Since $R_1\,C_1 = R_2\,C_2$ the last eqn can be written as:

$$V_{C_1}(s) = V_S(s)\,\frac{R_1}{R_1+R_2}$$

Taking the inverse Laplace transform we have

$$\begin{aligned} v_{C_1}(t) &= \frac{R_1}{R_1+R_2}\,v_S(t) \\ &= 2.5\,u(t) \end{aligned}$$

This shows that if $R_1\,C_1 = R_2\,C_2$ then the output waveform is a smaller though undistorted version of the input. This result is important for the design of attenuators and can be found in nearly all oscilloscope attenuators and probes.

Solution of problem 4.7

- *Circuit a):* Figure 4.13 shows the equivalent circuit in the s-domain. For this circuit we can

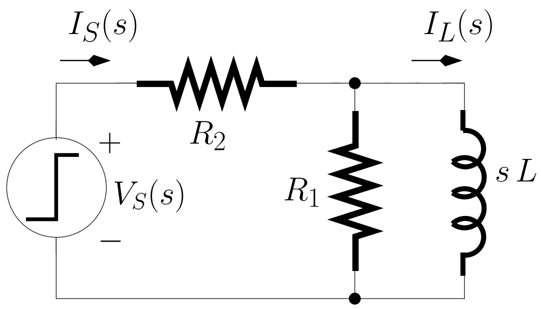


Figure 4.13: *Equivalent circuit in the s-domain.*

write:

$$\begin{aligned} I_S(s) &= \frac{V_S(s)}{R_2 + (R_1 || s\,L)} \\ &= V_S(s)\,\frac{R_1 + s\,L}{R_2\,R_1 + s\,L\,(R_2 + R_1)} \end{aligned}$$

Now the current in L can be obtained from the current divider expression:

$$\begin{aligned} I_L(s) &= I_S(s)\frac{R_1}{R_1 + s\,L} \\ &= V_S(s)\,\frac{R_1}{R_1 + R_2}\,\frac{1}{\frac{R_1\,R_2}{R_1+R_2} + s\,L} \\ &= \frac{V_s}{s}\,\frac{R_1}{R_1 + R_2}\,\frac{1}{\frac{R_1\,R_2}{R_1+R_2} + s\,L} \end{aligned}$$

with $V_s = 4$ V. Taking the inverse Laplace transform we obtain:

$$i_L(t) = \frac{V_s}{R_2}\left(1 - e^{-t/\tau}\right) \tag{4.7}$$

with $\tau = L/(R_1||R_2)$. Figure 4.14 shows $I_L(t)$.

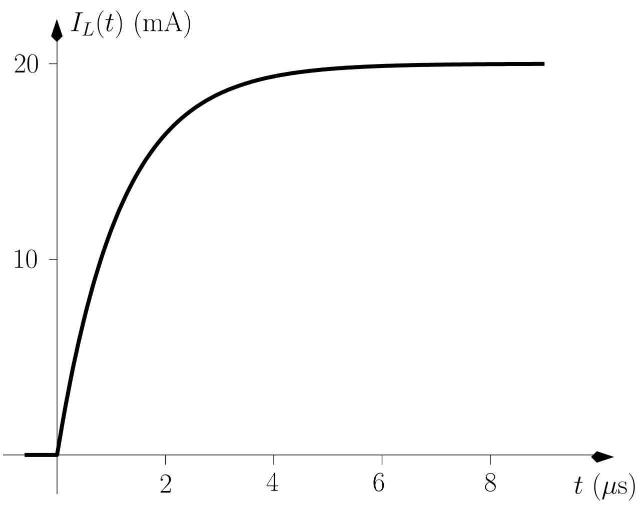


Figure 4.14: *The current $i_L(t)$.*

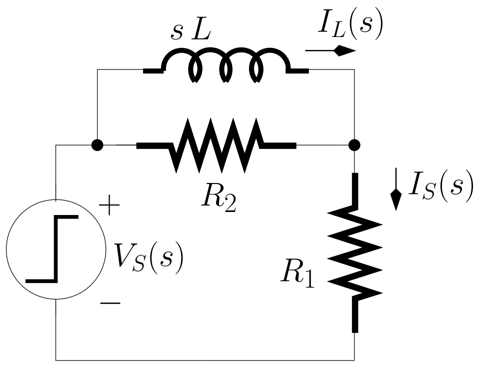


Figure 4.15: *Equivalent circuit in the s-domain.*

- *Circuit b):* Figure 4.15

shows the equivalent circuit in the s-domain. For this circuit we can write:

$$\begin{aligned} I_S(s) &= \frac{V_S(s)}{R_1 + (R_2 || s\,L)} \\ &= V_S(s)\,\frac{R_2 + s\,L}{R_2\,R_1 + s\,L\,(R_2 + R_1)} \end{aligned}$$

and

$$\begin{aligned} I_L(s) &= \frac{R_2}{R_2 + s\,L}\,I_L(s) \\ &= \frac{V_s}{s}\,\frac{1}{R_1}\,\frac{1}{1 + s\,L\,\frac{R_1+R_2}{R_1\,R_2}} \end{aligned}$$

Taking the inverse Laplace transform we obtain:

$$i_L(t) = \frac{V_s}{R_1}\left(1 - e^{-t/\tau}\right) \tag{4.8}$$

with $\tau = L/(R_1||R_2)$. Figure 4.16 shows $I_L(t)$.

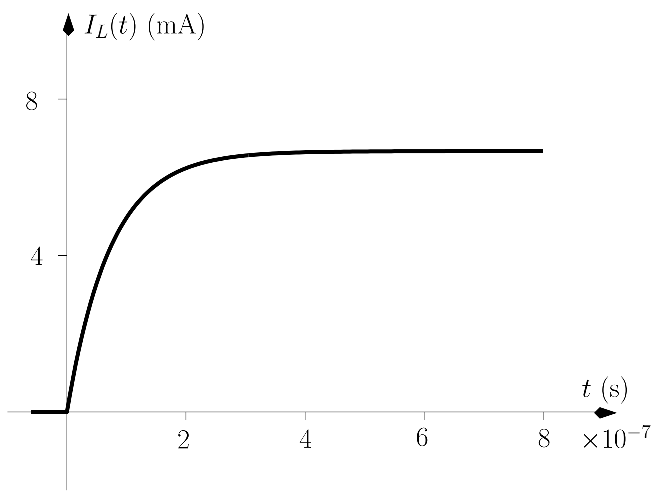


Figure 4.16: *The current $i_L(t)$.*

Solution of problem 4.8

Figure 4.17 shows the equivalent circuit in the s-domain. The current through the resistance can be

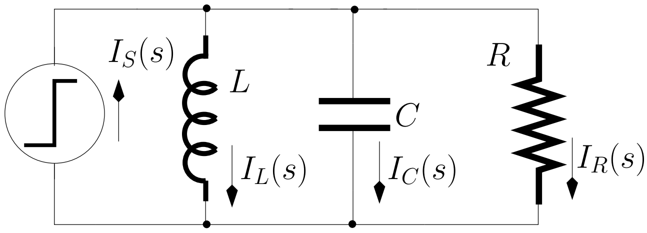


Figure 4.17: *Equivalent circuit in the s-domain.*

obtained as follows:

$$I_R(s) = I_S(s)\,\frac{Z_{LC}}{Z_{LC}+R}$$

with $I_S(s) = I_s/s$ and with Z_{LC} representing the parallel combination of $s\,L$ with $(s\,C)^{-1}$;

$$\begin{aligned} Z_{LC} &= s\,L||(s\,C)^{-1} \\ &= \frac{s\,L}{1+s^2\,L\,C} \end{aligned}$$

Hence, $I_R(s)$ can be written as:

$$\begin{aligned} I_R(s) &= I_S(s)\,\frac{s\,L}{s^2\,L\,C\,R+s\,L+R} \\ &= I_S(s)\,\frac{2\,\eta\,\omega_n\,s}{s^2+2\,\eta\,\omega_n\,s+\omega_n^2} \\ &= I_s\,\frac{2\,\eta\,\omega_n}{s^2+2\,\eta\,\omega_n\,s+\omega_n^2} \end{aligned}$$

with

$$\begin{aligned} \omega_n &= \frac{1}{\sqrt{L\,C}} \\ &= 64.6\ \text{krad/s} \\ \eta &= \frac{1}{2\,R}\sqrt{\frac{L}{C}} \\ &= 0.48 \end{aligned}$$

Taking the inverse Laplace transform we have:

$$i_R(t) = I_s\,\frac{2\,\eta\,e^{-t\,\eta\,\omega_n}}{\sqrt{1-\eta^2}}\,\sin\left(\sqrt{1-\eta^2}\omega_n\,t\right)\,u(t)$$

The current through the capacitor can be obtained as follows:

$$I_C(s) = I_S(s)\,\frac{Z_{LR}}{Z_{LR}+(s\,C)^{-1}}$$

with Z_{LR} representing the parallel combination of $s\,L$ and R;

$$\begin{aligned} Z_{LR} &= s\,L||R \\ &= \frac{s\,L\,R}{R+s\,L} \end{aligned}$$

Hence, $I_C(s)$ can be written as:

$$\begin{aligned} I_C(s) &= I_S(s)\,\frac{s^2\,L\,C\,R}{s^2\,L\,C\,R + s\,L + R} \\ &= I_S(s)\,\frac{s^2}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \\ &= I_s\,\frac{s}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \end{aligned}$$

Taking the inverse Laplace transform we have:

$$i_C(t) = I_s\,\frac{e^{-t\,\eta\,\omega_n}}{\sqrt{1-\eta^2}}\,\cos\left(\sqrt{1-\eta^2}\omega_n\,t + \phi\right)\,u(t)$$

with

$$\phi = \tan^{-1}\left(\frac{\eta}{\sqrt{1-\eta^2}}\right)$$

The current through the inductor can be obtained as:

$$I_L(s) = I_S(s)\,\frac{Z_{CR}}{Z_{CR} + s\,L}$$

with Z_{CR} representing the parallel combination of $(s\,C)^{-1}$ and R;

$$\begin{aligned} Z_{CR} &= (s\,C)^{-1}||R \\ &= \frac{R}{1 + s\,C\,R} \end{aligned}$$

Hence, $I_L(s)$ can be written as:

$$\begin{aligned} I_L(s) &= I_S(s)\,\frac{R}{s^2\,L\,C\,R + s\,L + R} \\ &= I_S(s)\,\frac{\omega_n^2}{s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2} \\ &= I_s\,\frac{\omega_n^2}{s\,(s^2 + 2\,\eta\,\omega_n\,s + \omega_n^2)} \end{aligned}$$

Taking the inverse Laplace transform we have:

$$i_L(t) = I_s\left[1 - \frac{e^{-t\,\eta\,\omega_n}}{\sqrt{1-\eta^2}}\,\sin\left(\sqrt{1-\eta^2}\omega_n\,t + \phi'\right)\right]u(t)$$

with

$$\phi' = \tan^{-1}\left(\frac{\sqrt{1-\eta^2}}{\eta}\right)$$

Figure 4.18 shows $i_S(t)$, $i_R(t)$, $i_C(t)$ and $i_L(t)$. Note the oscillatory response associated to $i_R(t)$, $i_C(t)$ and $i_L(t)$ due to $\eta < 1$. It is interesting to note that as $t \to \infty$ the inductor conducts I_s while R and C do not conduct. This is expected since, as $t \to \infty$ the electrical elements 'see' $i_S(t)$ as a DC current source and it is known that for DC the inductor behaves as a short-circuit and the capacitor behaves as an open-circuit.

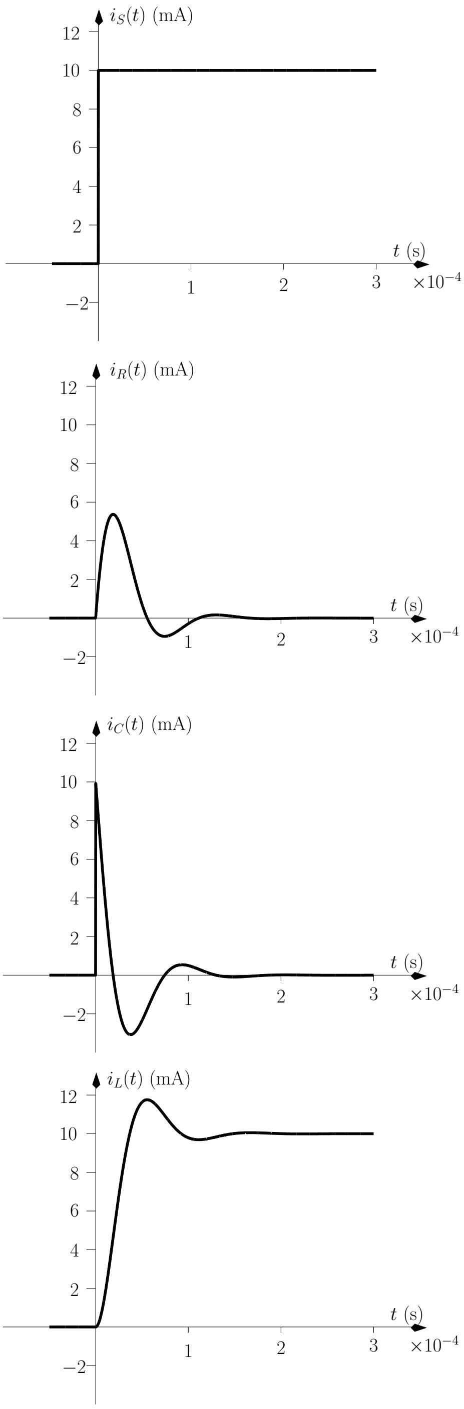


Figure 4.18: $i_S(t)$, $i_R(t)$, $i_C(t)$ *and* $i_L(t)$

Solution of problem 4.9

The circuit is critically damped when $\eta = 1$. Using the results obtained in the previous problem we have

$$\begin{aligned} R &= 2\sqrt{\frac{L}{C}} \\ &= 38.7\ \Omega \end{aligned}$$

Chapter 5

Electrical two-port network analysis

Solution of problem 5.1

- *Circuit a):* Figure 5.1 a) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. From this figure we can write:

$$\begin{aligned} V_1 &= (Z_1 + Z_2)\, I_1 \\ V_2 &= Z_2\, I_1 \end{aligned}$$

Hence we have:

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= Z_1 + Z_2 \\ Z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} \\ &= Z_2 \end{aligned}$$

Figure 5.1 b) shows the equivalent circuit for the calculation of Z_{12} and Z_{22}. For this circuit we can write:

$$\begin{aligned} V_2 &= Z_2\, I_2 \\ V_1 &= V_2 \end{aligned}$$

Hence we have:

$$\begin{aligned} Z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ &= Z_2 \\ Z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \\ &= Z_2 \end{aligned}$$

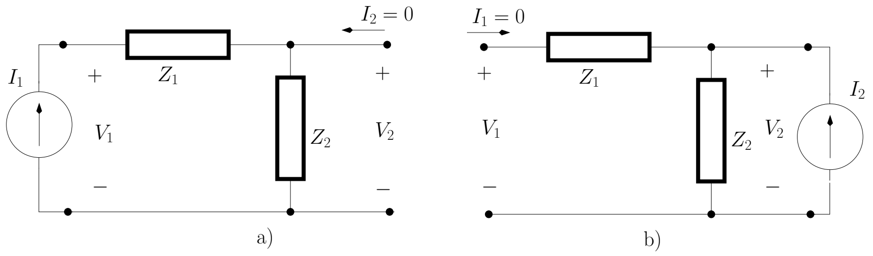


Figure 5.1: *a) Calculation of* Z_{11} *and* Z_{21}*. b) Calculation of* Z_{12} *and* Z_{22}*.*

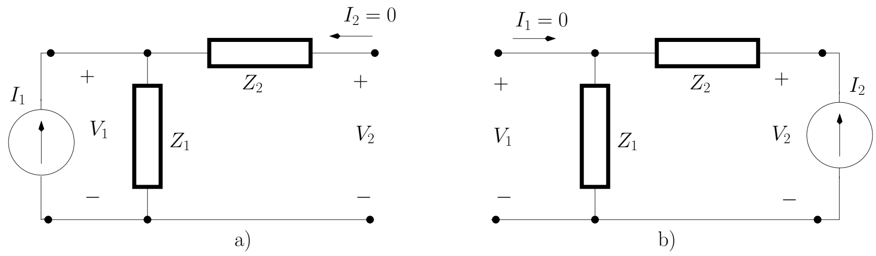


Figure 5.2: *a) Calculation of Z_{11} and Z_{21}. b) Calculation of Z_{12} and Z_{22}.*

- *Circuit b):* Figure 5.2 a) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. From this figure we can write:

$$\begin{aligned} V_1 &= Z_1 I_1 \\ V_2 &= V_1 \end{aligned}$$

Hence we have:

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= Z_1 \\ Z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} \\ &= Z_1 \end{aligned}$$

Figure 5.2 b) shows the equivalent circuit for the calculation of Z_{12} and Z_{22}. For this circuit we can write:

$$\begin{aligned} V_2 &= (Z_2 + Z_1)\, I_2 \\ V_1 &= Z_1 I_2 \end{aligned}$$

Hence we have:

$$\begin{aligned} Z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ &= Z_1 \\ Z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \\ &= Z_1 + Z_2 \end{aligned}$$

- *Circuit c):* Figure 5.3 a) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. From

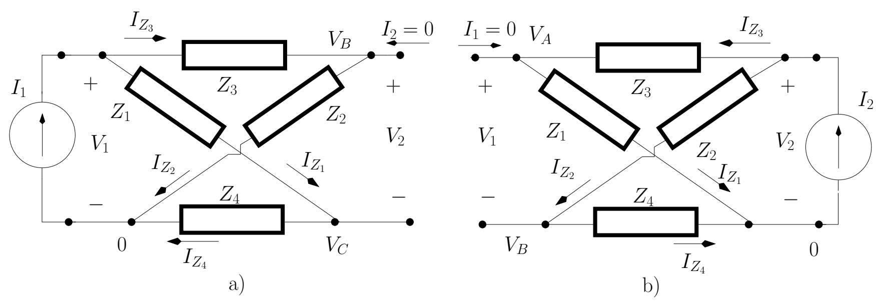


Figure 5.3: *a) Calculation of Z_{11} and Z_{21}. b) Calculation of Z_{12} and Z_{22}.*

this figure we can write:

$$\begin{cases} I_1 = I_{Z_1} + I_{Z_3} \\ I_1 = I_{Z_2} + I_{Z_4} \\ I_{Z_1} = I_{Z_4} \\ V_2 = V_B - V_C \end{cases} \tag{5.1}$$

that is

$$\begin{cases} I_1 = \frac{V_1 - V_C}{Z_1} + \frac{V_1 - V_B}{Z_3} \\ I_1 = \frac{V_B}{Z_2} + \frac{Z_C}{Z_4} \\ \frac{V_1 - V_C}{Z_1} = \frac{V_C}{Z_4} \\ V_2 = V_B - V_C \end{cases} \tag{5.2}$$

Solving, to obtain V_1 and V_2 we get:

$$\begin{aligned} V_1 &= I_1 \frac{(Z_3 + Z_2)(Z_4 + Z_1)}{Z_3 + Z_4 + Z_1 + Z_2} \\ V_2 &= I_1 \frac{(Z_2 Z_1 - Z_3 Z_4)}{Z_3 + Z_4 + Z_1 + Z_2} \end{aligned}$$

Hence we have:

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= \frac{(Z_3 + Z_2)(Z_4 + Z_1)}{Z_3 + Z_4 + Z_1 + Z_2} \\ Z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} \\ &= \frac{(Z_2 Z_1 - Z_3 Z_4)}{Z_3 + Z_4 + Z_1 + Z_2} \end{aligned}$$

Figure 5.3 b) shows the equivalent circuit for the calculation of Z_{12} and Z_{22}. For this circuit we can write:

$$\begin{cases} I_2 = I_{Z_3} + I_{Z_2} \\ I_2 = I_{Z_4} + I_{Z_1} \\ I_{Z_3} = I_{Z_1} \\ V_1 = V_A - V_B \end{cases} \tag{5.3}$$

that is

$$\begin{cases} I_2 = \frac{V_2 - V_A}{Z_3} + \frac{V_2 - V_B}{Z_2} \\ I_2 = \frac{V_B}{Z_4} + \frac{V_A}{Z_1} \\ \frac{V_2 - V_A}{Z_3} = \frac{V_A}{Z_1} \\ V_1 = V_A - V_B \end{cases} \tag{5.4}$$

Solving, to obtain V_1 and V_2 we get:

$$V_1 = I_2 \frac{Z_2 Z_1 - Z_3 Z_4}{Z_3 + Z_4 + Z_1 + Z_2}$$

$$V_2 = I_2 \frac{(Z_2+Z_4)(Z_3+Z_1)}{Z_3+Z_4+Z_1+Z_2}$$

Hence we have:

$$\begin{aligned} Z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ &= \frac{Z_2Z_1 - Z_3Z_4}{Z_3+Z_4+Z_1+Z_2} \\ Z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \\ &= \frac{(Z_2+Z_4)(Z_3+Z_1)}{Z_3+Z_4+Z_1+Z_2} \end{aligned}$$

- *Circuit d):* Figure 5.4 a) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. For

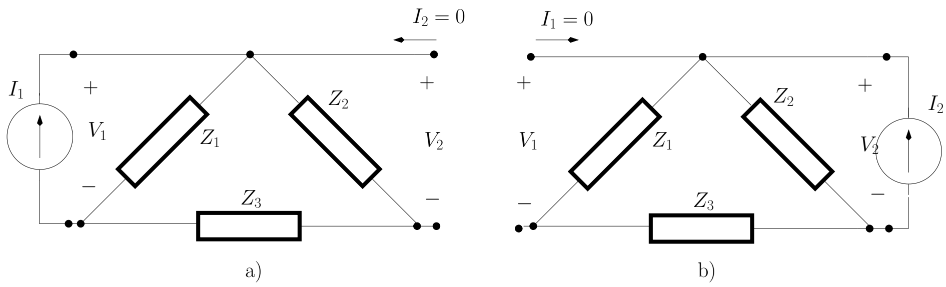


Figure 5.4: *a) Calculation of Z_{11} and Z_{21}. b) Calculation of Z_{12} and Z_{22}.*

this circuit we can write:

$$\begin{aligned} V_1 &= I_1 \left[Z_1 || (Z_2+Z_3)\right] \\ &= I_1 \frac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3} \end{aligned}$$

and

$$V_2 = I_1\, Z_2 \frac{Z_1}{Z_1+Z_2+Z_3}$$

Hence we have:

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= \frac{Z_1(Z_2+Z_3)}{Z_1+Z_2+Z_3} \\ Z_{22} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} \\ &= Z_2 \frac{Z_1}{Z_1+Z_2+Z_3} \end{aligned}$$

Figure 5.4 b) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. For this circuit we can write:

$$\begin{aligned} V_2 &= I_2\, Z_2 || (Z_1+Z_3) \\ &= I_2 \frac{Z_2\,(Z_1+Z_3)}{Z_1+Z_2+Z_3} \end{aligned}$$

and

$$V_1 = I_2\, Z_1 \frac{Z_2}{Z_1+Z_2+Z_3} \tag{5.5}$$

Hence we have:

$$\begin{aligned} Z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ &= Z_1\,\frac{Z_2}{Z_1+Z_2+Z_3} \\ Z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \\ &= \frac{Z_2\,(Z_1+Z_3)}{Z_1+Z_2+Z_3} \end{aligned}$$

- *Circuit e):* Figure 5.5 a) shows the equivalent circuit for the calculation of Z_{11} and Z_{21}. For

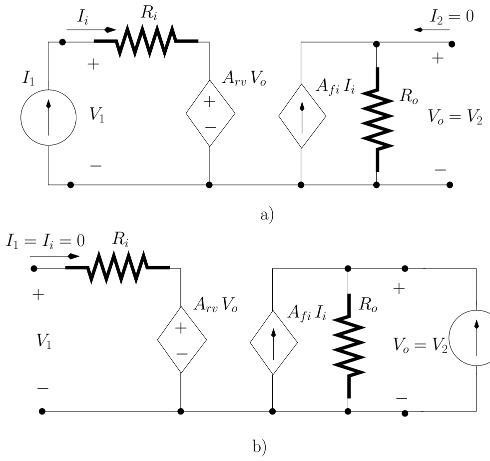


Figure 5.5: *a) Calculation of Z_{11} and Z_{21}. b) Calculation of Z_{12} and Z_{22}.*

this circuit we can write:

$$\begin{aligned} I_1 &= \frac{V_1 - A_{rv}\,V_o}{R_i} \\ &= \frac{V_1 - A_{rv}\,V_2}{R_i} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} V_o = V_2 &= A_{fi}\,I_i\,R_o \\ &= A_{fi}\,I_1\,R_o \end{aligned} \tag{5.7}$$

Equations 5.6 and 5.7 can be solved to obtain V_1 as follows:

$$V_1 = I_1\,(R_i + A_{rv}\,A_{fi}\,R_o)$$

Hence, we can write

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= R_i + A_{rv}\,A_{fi}\,R_o = 103\ \text{k}\Omega \\ Z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} \\ &= A_{fi}\,R_o = 1\ \text{M}\Omega \end{aligned}$$

Figure 5.5 b) shows the equivalent circuit for the calculation of Z_{12} and Z_{22}. For this circuit we have $I_1 = I_i = 0$. Hence, we can write:

$$V_2 = I_2\,R_o$$

and

$$\begin{aligned} V_1 &= A_{rv}\, V_2 \\ &= A_{rv}\, I_2\, R_o \end{aligned}$$

Finally we have,

$$\begin{aligned} Z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\ &= A_{rv}\, R_o = 1 \ \text{k}\Omega \\ Z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \\ &= R_o = 10 \ \text{k}\Omega \end{aligned}$$

Solution of problem 5.2

Figure 5.6 a) shows the circuit for the calculation of $Z_{eq_{11}}$ and $Z_{eq_{21}}$. $Z_{eq_{11}}$ can be determined as follows:

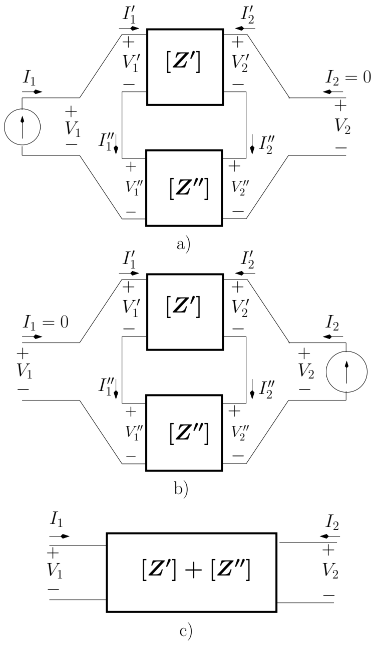


Figure 5.6: *a) Calculation of $Z_{eq_{11}}$ and $Z_{eq_{21}}$. b) Calculation of $Z_{eq_{12}}$ and $Z_{eq_{22}}$. c) Equivalent two-port circuit.*

$$Z_{eq_{11}} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

Since port 2 is an open-circuit then $I_2 = I_2' = I_2'' = 0$ and $I_1 = I_1' = I_1''$. We also have $V_1 = V_1' + V_1''$. Therefore, we can write the last eqn as follows:

$$\begin{aligned} Z_{eq_{11}} &= \left.\frac{V_1'}{I_1'}\right|_{I_2'=0} + \left.\frac{V_1''}{I_1''}\right|_{I_2''=0} \\ &= Z_{11}' + Z_{11}'' \end{aligned}$$

In addition we have $V_2 = V_2' + V_2''$. Therefore we can write:

$$\begin{aligned} Z_{eq_{21}} &= \left.\frac{V_2'}{I_1'}\right|_{I_2'=0} + \left.\frac{V_2''}{I_1''}\right|_{I_2''=0} \\ &= Z_{21}' + Z_{21}'' \end{aligned}$$

Figure 5.6 b) shows the circuit for the calculation of $Z_{eq_{12}}$ and $Z_{eq_{22}}$. $Z_{eq_{12}}$ is given by:

$$Z_{eq_{12}} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

Since port 1 is an open-circuit then $I_1 = I_1' = I_1'' = 0$ and $I_2 = I_2' = I_2''$. Again we have $V_1 = V_1' + V_1''$ and $V_2 = V_2' + V_2''$. Therefore, we can write:

$$\begin{aligned} Z_{eq_{12}} &= \left.\frac{V_1'}{I_2'}\right|_{I_1'=0} + \left.\frac{V_1''}{I_2''}\right|_{I_1''=0} \\ &= Z_{12}' + Z_{12}'' \end{aligned}$$

and

$$\begin{aligned} Z_{eq_{22}} &= \left.\frac{V_2'}{I_2'}\right|_{I_1'=0} + \left.\frac{V_2''}{I_2''}\right|_{I_1''=0} \\ &= Z_{22}' + Z_{22}'' \end{aligned}$$

From the above we can write

$$[\boldsymbol{Z_{eq}}] = [\boldsymbol{Z'}] + [\boldsymbol{Z''}]$$

Solution of problem 5.3

- *Circuit a):* Figure 5.7 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}. For

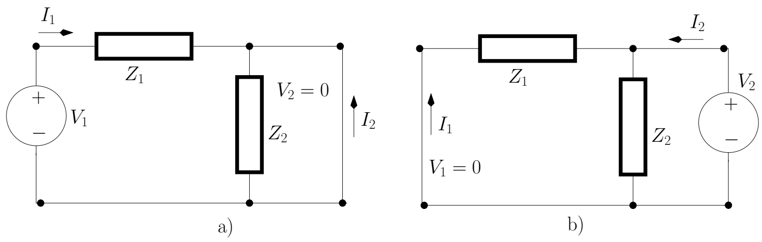


Figure 5.7: *a) Calculation of Y_{11} and Y_{21}. b) Calculation of Y_{12} and Y_{22}.*

this circuit we can write:

$$I_1 = V_1 \frac{1}{Z_1}$$

and

$$I_2 = V_1 \frac{-1}{Z_1} = -I_1$$

Hence we can write:

$$\begin{aligned} Y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= \frac{1}{Z_1} \\ Y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\ &= \frac{-1}{Z_1} \end{aligned}$$

Figure 5.7 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}. For this circuit we can write:

$$\begin{aligned} I_2 &= V_2 \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) \\ I_2 &= V_2 \left(Y_1 + Y_2\right) \end{aligned}$$

and

$$I_1 = V_2 \frac{-1}{Z_1}$$

Hence we can write:

$$\begin{aligned} Y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ &= \frac{-1}{Z_1} \\ Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= Y_1 + Y_2 \end{aligned}$$

- *Circuit b):* Figure 5.8 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}. For

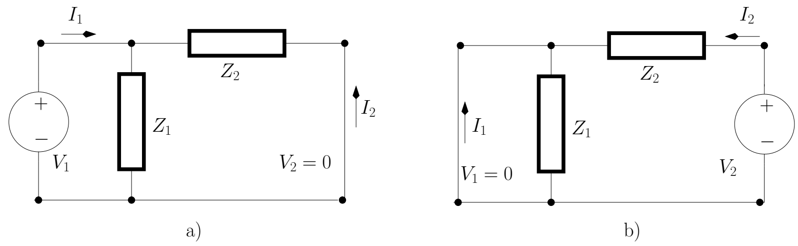


Figure 5.8: *a) Calculation of* Y_{11} *and* Y_{21}*. b) Calculation of* Y_{12} *and* Y_{22}.

this circuit we can write:

$$\begin{aligned} Y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= Y_1 + Y_2 \\ Y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\ &= -Y_2 \end{aligned}$$

Figure 5.8 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}. For this circuit we can write:

$$\begin{aligned} Y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ &= -Y_2 \\ Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= Y_2 \end{aligned}$$

- *Circuit c):* Figure 5.9 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}. For

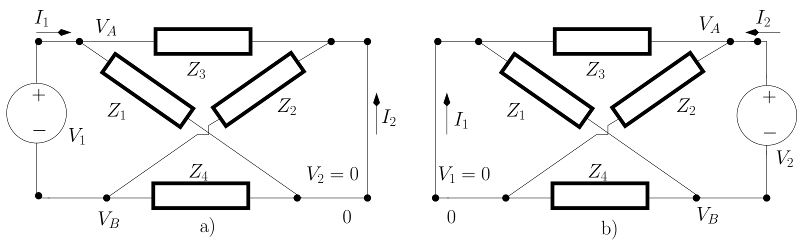


Figure 5.9: *a) Calculation of* Y_{11} *and* Y_{21}*. b) Calculation of* Y_{12} *and* Y_{22}.

this circuit we can write:

$$\begin{cases} V_A - V_B = V_1 \\ V_A\, Y_1 + V_A\, Y_3 = I_1 \\ -V_B\, Y_2 - V_B\, Y_4 = I_1 \\ V_A\, Y_1 + V_B\, Y_4 = I_2 \\ -V_A\, Y_3 - V_B\, Y_2 = I_2 \end{cases} \tag{5.8}$$

Solving to obtain I_1 and I_2 we get:

$$\begin{aligned} I_1 &= V_1\, \frac{(Y_1 + Y_3)\,(Y_2 + Y_4)}{Y_1 + Y_3 + Y_2 + Y_4} \\ I_2 &= -V_1 \frac{Y_3\, Y_4 - Y_2\, Y_1}{Y_1 + Y_4 + Y_3 + Y_2} \end{aligned}$$

that is

$$\begin{aligned} Y_{11} &= \frac{(Y_1+Y_3)\,(Y_2+Y_4)}{Y_1+Y_3+Y_2+Y_4} \\ Y_{21} &= \frac{Y_2\,Y_1 - Y_3\,Y_4}{Y_1+Y_4+Y_3+Y_2} \end{aligned}$$

Figure 5.9 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}. For this circuit we can write:

$$\begin{cases} V_A - V_B = V_2 \\ V_A\,Y_2 + V_A\,Y_3 = I_2 \\ -V_B\,Y_1 - V_B\,Y_4 = I_2 \\ V_A\,Y_2 + V_B\,Y_4 = I_1 \\ -V_A\,Y_3 - V_B\,Y_1 = I_1 \end{cases} \tag{5.9}$$

Solving to obtain I_1 and I_2 we get:

$$\begin{aligned} I_1 &= -V_2\,\frac{Y_3\,Y_4 - Y_2\,Y_1}{Y_1+Y_4+Y_3+Y_2} \\ I_2 &= V_2\,\frac{(Y_3+Y_2)\,(Y_1+Y_4)}{Y_1+Y_4+Y_3+Y_2} \end{aligned}$$

that is:

$$\begin{aligned} Y_{12} &= \frac{Y_2\,Y_1 - Y_3\,Y_4}{Y_1+Y_4+Y_3+Y_2} \\ Y_{22} &= \frac{(Y_3+Y_2)\,(Y_1+Y_4)}{Y_1+Y_4+Y_3+Y_2} \end{aligned}$$

Note the symmetrical results as expected.

- *Circuit d):* Figure 5.10 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}. For

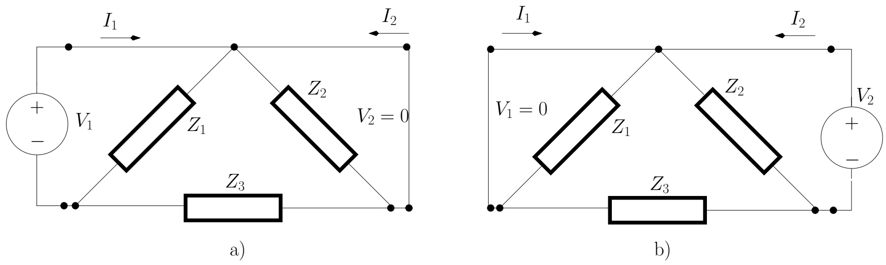


Figure 5.10: *a) Calculation of* Y_{11} *and* Y_{21}. *b) Calculation of* Y_{12} *and* Y_{22}.

this circuit we can write:

$$\begin{aligned} I_1 &= V_1\,(Y_1+Y_3) \\ I_2 &= -V_1\,Y_3 \end{aligned}$$

Hence, we can write:

$$\begin{aligned} Y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= Y_1 + Y_3 \\ Y_{21} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= -Y_3 \end{aligned}$$

Figure 5.10 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}. For this circuit we can write:

$$\begin{aligned} I_1 &= -V_2\, Y_3 \\ I_2 &= V_2\,(Y_2 + Y_3) \end{aligned}$$

Hence, we can write:

$$\begin{aligned} Y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ &= -Y_3 \\ Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= Y_2 + Y_3 \end{aligned}$$

- *Circuit e):* Figure 5.11 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}. For

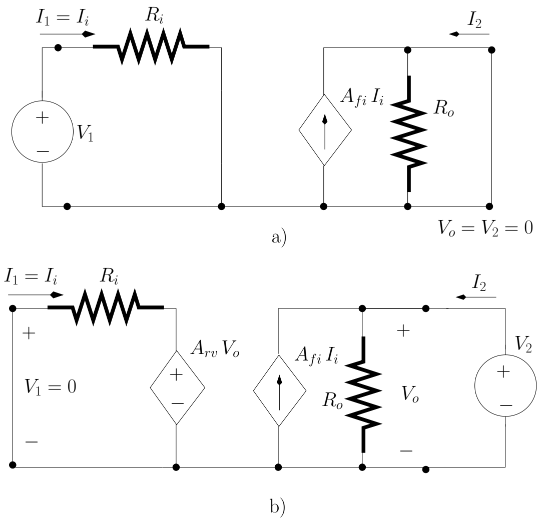


Figure 5.11: *Calculation of: a)* Y_{11}*; b)* Y_{21}.

this circuit we can write:

$$\begin{aligned} V_1 &= R_i\, I_1 \\ I_2 &= A_i\, \frac{V_1}{R_i} \end{aligned}$$

Hence we can write:

$$\begin{aligned} Y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= \frac{1}{R_i} = 0.33\ \text{mS} \end{aligned}$$

and

$$\begin{aligned} Y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\ &= \frac{A_{fi}}{R_i} = 33.33\ \text{mS} \end{aligned}$$

Figure 5.11 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}. For this circuit we can write:

$$I_1 = -\frac{A_{rv}\, V_2}{R_i}$$

$$I_2 = \frac{V_2}{R_o} + \frac{A_{rv}\,A_{fi}\,V_2}{R_i}$$

Hence we can write:

$$\begin{aligned} Y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ &= -\frac{A_{rv}}{R_i} = -33.33\ \mu\text{S} \end{aligned}$$

and

$$\begin{aligned} Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= \frac{1}{R_o} + \frac{A_{rv}\,A_{fi}}{R_i} = 3.4\ \text{mS} \end{aligned}$$

Solution of problem 5.4

Figure 5.12 a) shows the circuit for the calculation of $Y_{eq_{11}}$ and $Y_{eq_{21}}$. $Y_{eq_{11}}$ can be determined as follows:

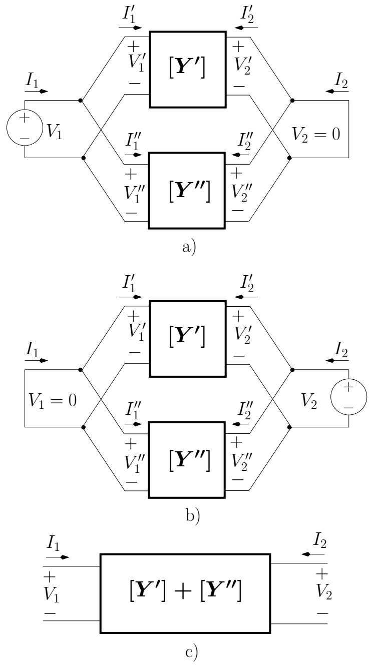


Figure 5.12: *a) Calculation of $Y_{eq_{11}}$ and $Y_{eq_{21}}$. b) Calculation of $Y_{eq_{12}}$ and $Y_{eq_{22}}$. c) Equivalent two-port circuit.*

$$Y_{eq_{11}} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

Since port 2 is short-circuited $V_2 = V_2' = V_2'' = 0$. In addition we have $V_1 = V_1' = V_1''$, $I_2 = I_2' + I_2''$ and $I_1 = I_1' + I_1''$. Therefore, we can rewrite the last eqn as follows:

$$\begin{aligned} Y_{eq_{11}} &= \left.\frac{I_1'}{V_1'}\right|_{V_2'=0} + \left.\frac{I''}{V_1''}\right|_{V_2''=0} \\ &= Y_{11}' + Y_{11}'' \end{aligned}$$

and $Y_{eq_{21}}$ can be determined as:

$$\begin{aligned} Y_{eq_{21}} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\ &= \left.\frac{I_2'}{V_1'}\right|_{V_2'=0} + \left.\frac{I_2''}{V_1''}\right|_{V_2''=0} \\ &= Y_{21}' + Y_{21}'' \end{aligned}$$

Figure 5.12 b) shows the circuit for the calculation of $Y_{eq_{11}}$ and $Y_{eq_{21}}$. These can be determined as follows:

$$\begin{aligned} Y_{eq_{12}} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ Y_{eq_{22}} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \end{aligned}$$

Since port 1 is short-circuited, $V_1 = V_1' = V_1'' = 0$. We also have $V_2 = V_2' = V_2''$, $I_2 = I_2' + I_2''$ and $I_1 = I_1' + I_1''$. Therefore, we can rewrite the last two eqns as follows:

$$\begin{aligned}
Y_{eq_{12}} &= \left.\frac{I_1'}{V_2'}\right|_{V_1'=0} + \left.\frac{I_1''}{V_2''}\right|_{V_1''=0} \\
&= Y_{12}' + Y_{21}'' \\
Y_{eq_{22}} &= \left.\frac{I_2'}{V_2'}\right|_{V_1'=0} + \left.\frac{I_2''}{V_2''}\right|_{V_1''=0} \\
&= Y_{22}' + Y_{22}''
\end{aligned}$$

From the above we can see that

$$[\boldsymbol{Y_{eq}}] = [\boldsymbol{Y'}] + [\boldsymbol{Y''}]$$

Solution of problem 5.5

- *Circuit a):* Figure 5.13 a) shows the equivalent circuit for the calculation of A_{11}. For this

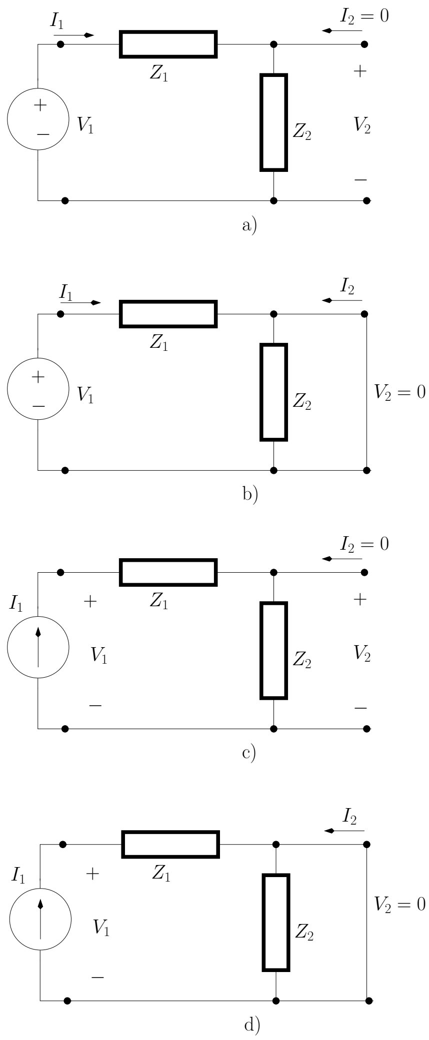


Figure 5.13: *Calculation of: a) A_{11}; b) A_{12}; c) A_{21}; d) A_{22}.*

circuit we can write:

$$V_2 = V_1 \frac{Z_2}{Z_2 + Z_1}$$

Hence,

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \frac{Z_2 + Z_1}{Z_2} \end{aligned}$$

Figure 5.13 b) shows the equivalent circuit for the calculation of A_{12}. For this circuit we can write:

$$V_1 = -I_2\, Z_1$$

Hence,

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= Z_1 \end{aligned}$$

Figure 5.13 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit we can write:

$$V_2 = I_1 Z_2$$

Hence,

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{1}{Z_2} \end{aligned}$$

Figure 5.13 d) shows the equivalent circuit for the calculation of A_{22}. For this circuit we can write:

$$I_1 = -I_2$$

Hence,

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= 1 \end{aligned}$$

- *Circuit b):* Figure 5.14 a) shows the equivalent circuit for the calculation of A_{11}. For this

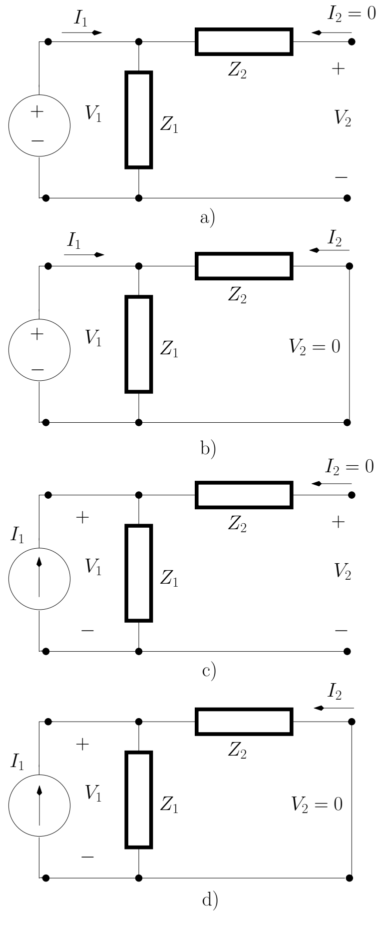


Figure 5.14: *Calculation of: a) A_{11}; b) A_{12}; c) A_{21}; d) A_{22}.*

circuit we write:

$$V_1 = V_2$$

that is:

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= 1 \end{aligned}$$

Figure 5.14 b) shows the equivalent circuit for the calculation of A_{12}. For this circuit we can write:

$$V_1 = -I_2\, Z_2$$

Hence,

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= Z_2 \end{aligned}$$

Figure 5.14 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit, since $I_2 = 0$, we can write:

$$V_2 = I_1\, Z_1$$

Hence,

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{1}{Z_1} \end{aligned}$$

Figure 5.14 d) shows the equivalent circuit for the calculation of A_{22}. For this circuit we can write:

$$I_2 = -I_1 \frac{Z_1}{Z_1 + Z_2}$$

Hence,

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{Z_1 + Z_2}{Z_1} \end{aligned}$$

- *Circuit c):* Figure 5.15 a) shows the equivalent circuit for the calculation of A_{11}. For this circuit we can write the following set of eqns:

$$\begin{cases} V_1 = V_A - V_B \\ I_{Z_3} = I_{Z_2} \\ I_{Z_1} = I_{Z_4} \end{cases} \tag{5.10}$$

that is

$$\begin{cases} V_1 = V_A - V_B \\ \dfrac{V_A - V_2}{Z_3} = \dfrac{V_2 - V_B}{Z_2} \\ \dfrac{V_A}{Z_1} = -\dfrac{V_B}{Z_4} \end{cases} \tag{5.11}$$

Solving to obtain V_1 we get

$$V_1 = V_2 \frac{(Z_2 + Z_3)\,(Z_1 + Z_4)}{Z_2\, Z_1 - Z_3\, Z_4}$$

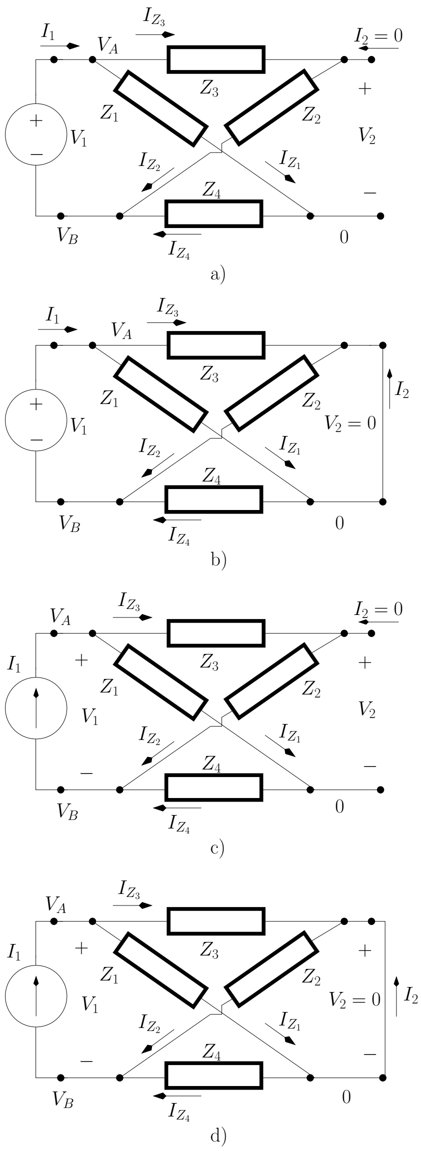


Figure 5.15: *Calculation of: a) A_{11}; b) A_{12}; c) A_{21}; d) A_{22}.*

that is:

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \frac{(Z_2+Z_3)\,(Z_1+Z_4)}{Z_2\,Z_1 - Z_3\,Z_4} \end{aligned}$$

Figure 5.15 b) shows the equivalent circuit for the calculation of A_{12}. For this circuit we can write the following set of eqns:

$$\begin{cases} V_1 = V_A - V_B \\ I_{Z_3} - I_{Z_2} = -I_2 \\ I_{Z_4} - I_{Z_1} = -I_2 \end{cases} \tag{5.12}$$

that is

$$\begin{cases} V_1 = V_A - V_B \\ \dfrac{V_A}{Z_3} + \dfrac{V_B}{Z_2} = -I_2 \\ -\dfrac{V_B}{Z_4} - \dfrac{V_A}{Z_1} = -I_2 \end{cases} \tag{5.13}$$

Solving to obtain V_1 we get

$$V_1 = I_2 \frac{Z_1\,Z_3\,Z_4 + Z_1\,Z_3\,Z_2 + Z_1\,Z_4\,Z_2 + Z_3\,Z_4\,Z_2}{Z_3\,Z_4 - Z_2\,Z_1}$$

that is:

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= \frac{Z_1\,Z_3\,Z_4 + Z_1\,Z_3\,Z_2 + Z_1\,Z_4\,Z_2 + Z_3\,Z_4\,Z_2}{Z_2\,Z_1 - Z_3\,Z_4} \end{aligned}$$

Figure 5.15 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit we can write:

$$\begin{cases} I_1 = I_{Z_3} + I_{Z_1} \\ I_1 = I_{Z_2} + I_{Z_4} \\ I_{Z_3} = I_{Z_2} \end{cases} \tag{5.14}$$

that is

$$\begin{cases} I_1 = \dfrac{V_A - V_2}{Z_3} + \dfrac{V_A}{Z_1} \\ I_1 = \dfrac{V_2 - V_B}{Z_2} - \dfrac{V_B}{Z_4} \\ \dfrac{V_A - V_2}{Z_3} = \dfrac{V_2 - V_B}{Z_2} \end{cases} \tag{5.15}$$

Solving to obtain I_1 we get

$$I_1 = V_2 \frac{Z_1 + Z_4 + Z_2 + Z_3}{Z_2\,Z_1 - Z_3\,Z_4}$$

and

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{Z_1 + Z_4 + Z_2 + Z_3}{Z_2\,Z_1 - Z_3\,Z_4} \end{aligned}$$

Figure 5.15 d) shows the equivalent circuit for the calculation of A_{22}. For this circuit we can write:

$$\begin{cases} I_1 = I_{Z_3} + I_{Z_1} \\ I_1 = I_{Z_2} + I_{Z_4} \\ -I_2 = I_{Z_3} - I_{Z_2} \end{cases} \tag{5.16}$$

that is

$$\begin{cases} I_1 = \dfrac{V_A}{Z_3} + \dfrac{V_A}{Z_1} \\ I_1 = \dfrac{-V_B}{Z_2} - \dfrac{V_B}{Z_4} \\ -I_2 = \dfrac{V_A}{Z_3} + \dfrac{V_B}{Z_2} \end{cases} \tag{5.17}$$

Solving to obtain I_1 we can write

$$I_1 = I_2\,\frac{(Z_1+Z_3)(Z_4+Z_2)}{Z_3\,Z_4 - Z_2\,Z_1}$$

and finally,

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{(Z_1+Z_3)(Z_4+Z_2)}{Z_2\,Z_1 - Z_3\,Z_4} \end{aligned}$$

- *Circuit d):* Figure 5.16 a) shows the equivalent circuit for the calculation of A_{11}. For this

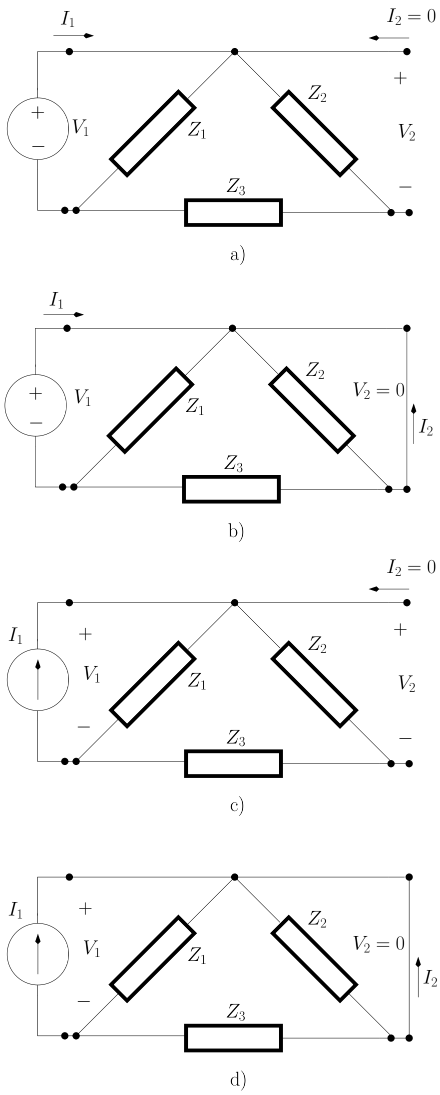


Figure 5.16: *Calculation of: a) A_{11}; b) A_{12}; c) A_{21}; d) A_{22}.*

circuit we can write:

$$V_2 = V_1\,\frac{Z_2}{Z_2+Z_3}$$

that is

$$A_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

$$= \frac{Z_2}{Z_2 + Z_3}$$

Figure 5.16 b) shows the equivalent circuit for the calculation of A_{12}. For this circuit we can write:

$$V_1 = -I_2\, Z_3$$

that is:

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= Z_3 \end{aligned}$$

Figure 5.16 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit we can write:

$$V_2 = I_1\, \frac{Z_2\, Z_1}{Z_1 + Z_2 + Z_3}$$

that is:

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{Z_1 + Z_2 + Z_3}{Z_2\, Z_1} \end{aligned}$$

Figure 5.16 d) shows the equivalent circuit for the calculation of A_{22}. For this circuit we can write:

$$-I_2 = I_1\, \frac{Z_1}{Z_1 + Z_3}$$

that is

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{Z_1 + Z_3}{Z_1} \end{aligned}$$

- *Circuit e):* Figure 5.17 a) shows the equivalent circuit for the calculation of A_{11}. For this circuit we can write:

$$\begin{aligned} V_2 &= A_{fi}\, I_i\, R_o \\ I_1 = I_i &= \frac{V_1 - A_{rv}\, V_2}{R_i} \end{aligned}$$

Hence we can write V_2 as follows:

$$V_2 = A_{fi}\, R_o\, \frac{V_1 - A_{rv}\, V_2}{R_i}$$

that is

$$V_2 = V_1\, \frac{A_{fi}\, R_o}{R_i + A_{rv}\, A_{fi}\, R_o}$$

and

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \left(\frac{A_{fi}\, R_o}{R_i + A_{rv}\, A_{fi}\, R_o}\right)^{-1} = 0.1 \end{aligned}$$

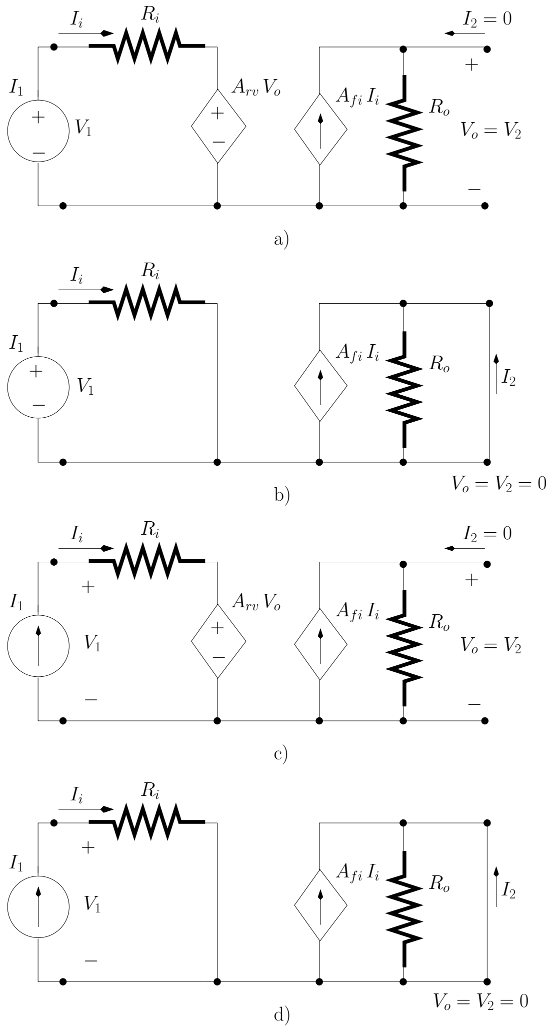


Figure 5.17: *Calculation of: a) A_{11}; b) A_{12}; c) A_{21}; d) A_{22}.*

Figure 5.17 b) shows the equivalent circuit for the calculation of A_{11}. Note that since $V_o = V_2 = 0$, the voltage controlled-current source is replaced by a short-circuit. For this circuit we can write:

$$\begin{aligned} -I_2 &= A_{fi}\, I_i \\ &= A_{fi}\, \frac{V_1}{R_i} \end{aligned}$$

that is

$$\begin{aligned} A_{12} &= \left. \frac{V_1}{-I_2} \right|_{V_2=0} \\ &= \frac{R_i}{A_{fi}} = 30\ \Omega \end{aligned}$$

Figure 5.17 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit we can write:

$$V_2 = A_{fi}\, I_1\, R_o$$

and

$$A_{21} = \left. \frac{I_1}{V_2} \right|_{I_2=0}$$

$$= \quad (A_{fi}\, R_o)^{-1} = 1 \ \ \mu\text{S}$$

Figure 5.17 c) shows the equivalent circuit for the calculation of A_{21}. For this circuit we can write:

$$\begin{aligned} A_{22} &= \left. \frac{I_1}{-I_2} \right|_{V_2=0} \\ &= \frac{1}{A_{fi}} = 0.01 \end{aligned}$$

Solution of problem 5.6

The chain parameters for each two port circuit of figure 5.18 satisfy the following eqns:

$$V_1' = A_{11}'\,V_2' - A_{12}'\,I_2' \tag{5.18}$$
$$I_1' = A_{21}'\,V_2' - A_{22}'\,I_2' \tag{5.19}$$

and

$$V_1'' = A_{11}''\,V_2'' - A_{12}''\,I_2'' \tag{5.20}$$
$$I_1'' = A_{21}''\,V_2'' - A_{22}''\,I_2'' \tag{5.21}$$

Figure 5.18 a) shows the circuit for the calculation of $A_{eq_{11}}$ which is given by

$$A_{eq_{11}} = \left.\frac{V_1}{V_2}\right|_{I_2=0} \tag{5.22}$$

Using eqn 5.18 and since $V_1 = V_1'$, $V_2' = V_1''$, $V_2 = V_2''$ and $I_2' = -I_1''$ we can write eqn 5.22 as follows:

$$A_{eq_{11}} = A_{11}'\left.\frac{V_1''}{V_2''}\right|_{I_2''=0} + A_{12}'\left.\frac{I_1''}{V_2''}\right|_{I_2''=0} \tag{5.23}$$

that is:

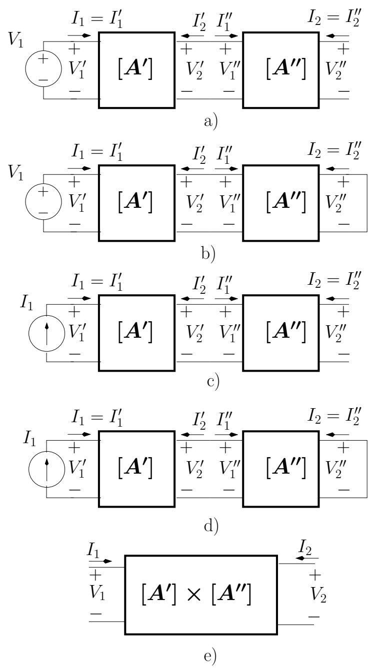


Figure 5.18: *a) Calculation of $A_{eq_{11}}$. b) Calculation of $Y_{eq_{21}}$. c) Calculation of $Y_{eq_{12}}$. d) $Y_{eq_{22}}$. e) Equivalent two-port circuit.*

$$A_{eq_{11}} = A_{11}'\,A_{11}'' + A_{12}'\,A_{21}''$$

Figure 5.18 b) shows the circuit for the calculation of $A_{eq_{12}}$ which is given by

$$A_{eq_{12}} = \left.\frac{V_1}{-I_2}\right|_{V_2=0} \tag{5.24}$$

Using again the result of eqn 5.18 and since $V_1 = V_1'$, $V_2' = V_1''$, $V_2 = V_2'' = 0$ and $I_2' = -I_1''$ we can write eqn 5.24 as follows:

$$\begin{aligned} A_{eq_{12}} &= A_{11}' \left. \frac{V_1''}{-I_2''} \right|_{V_2''=0} + A_{12}' \left. \frac{I_1''}{-I_2''} \right|_{V_2''=0} \\ &= A_{11}' \, A_{12}'' + A_{12}' \, A_{22}'' \end{aligned}$$

Figure 5.18 c) shows the circuit for the calculation of $A_{eq_{21}}$ which is given by

$$A_{eq_{21}} = \left. \frac{I_1}{V_2} \right|_{I_2=0} \tag{5.25}$$

Using the result of eqn 5.19 and since $V_1 = V_1'$, $I_1 = I_1'$, $V_2' = V_1''$, $I_2 = I_2'' = 0$ and $I_2' = -I_1''$ we can write eqn 5.25 as follows:

$$\begin{aligned} A_{eq_{21}} &= A_{21}' \left. \frac{V_1''}{V_2''} \right|_{I_2''=0} + A_{22}' \left. \frac{I_1''}{V_2''} \right|_{I_2''=0} \\ &= A_{21}' \, A_{11}'' + A_{22}' \, A_{21}'' \end{aligned}$$

Figure 5.18 d) shows the circuit for the calculation of $A_{eq_{22}}$ which is given by

$$A_{eq_{22}} = \left. \frac{I_1}{-I_2} \right|_{V_2=0} \tag{5.26}$$

Using the result of eqn 5.19 and since $V_1 = V_1'$, $I_1 = I_1'$, $V_2' = V_1''$, $V_2 = V_2'' = 0$ and $I_2' = -I_1''$ we can write eqn 5.26 as follows:

$$\begin{aligned} A_{eq_{22}} &= A_{21}' \left. \frac{V_1''}{-I_2''} \right|_{V_2''=0} + A_{22}' \left. \frac{I_1''}{-I_2''} \right|_{V_2''=0} \\ &= A_{21}' \, A_{12}'' + A_{22}' \, A_{22}'' \end{aligned}$$

From the above we get:

$$[\boldsymbol{A_{eq}}] = [\boldsymbol{A'}] \times [\boldsymbol{A''}] \tag{5.27}$$

Solution of problem 5.7

$$[\boldsymbol{V}] \quad = \quad [\boldsymbol{Z}]\,[\boldsymbol{I}]$$

Using elementary matrix algebra we can solve the last eqn to obtain $[\boldsymbol{Y}]$

$$[\boldsymbol{Y}] \quad = \quad [\boldsymbol{Z}]^{-1}\,[\boldsymbol{V}]$$

that is $[\boldsymbol{Y}] = [\boldsymbol{Z}]^{-1}$.

Solution of problem 5.8

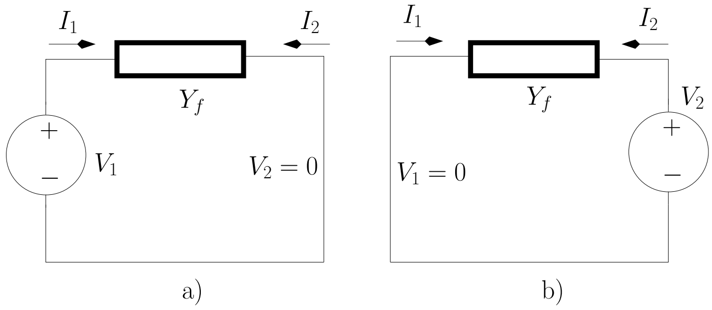


Figure 5.19: *a) Calculation of* Y_{11} *and* Y_{21}. *b) Calculation of* Y_{12} *and* Y_{22}.

Figure 5.19 a) shows the equivalent circuit for the calculation of Y_{11} and Y_{21}

$$\begin{aligned} Y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\ &= Y_f \\ Y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\ &= -Y_f \end{aligned}$$

Figure 5.19 b) shows the equivalent circuit for the calculation of Y_{12} and Y_{22}

$$\begin{aligned} Y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ &= -Y_f \\ Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= Y_f \end{aligned}$$

Solution of problem 5.9

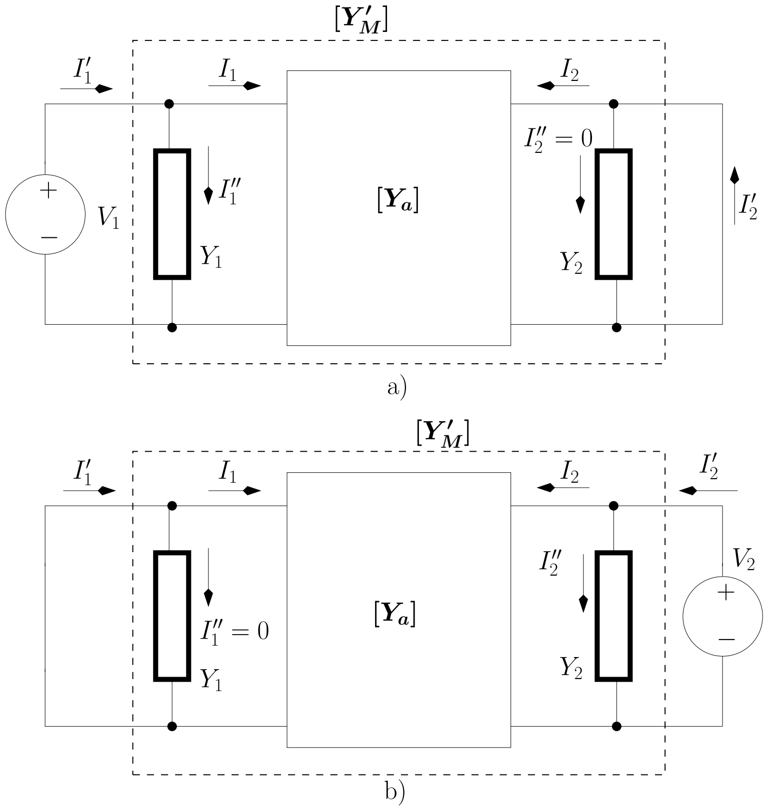


Figure 5.20: *a) Calculation of* $Y_{M_{11}}$ *and* $Y_{M_{21}}$. *b) Calculation of* $Y_{M_{12}}$ *and* $Y_{M_{22}}$.

Figure 5.20 a) shows the equivalent circuit for the calculation of $Y_{M_{11}}$ and $Y_{M_{12}}$. $Y_{M_{11}}$ is given by;

$$\begin{aligned}
Y_{M_{11}} &= \left.\frac{I'_1}{V_1}\right|_{V_2=0} \\
&= \left.\frac{I_1 + I''_1}{V_1}\right|_{V_2=0} \\
&= \left.\frac{I_1 + Y_1\, V_1}{V_1}\right|_{V_2=0} \\
&= \left.\frac{I_1}{V_1}\right|_{V_2=0} + Y_1 \\
&= Y_{a_{11}} + Y_1
\end{aligned}$$

$Y_{M_{21}}$ is given by;

$$\begin{aligned}
Y_{M_{21}} &= \left.\frac{I'_2}{V_1}\right|_{V_2=0} \\
&= \left.\frac{I_2}{V_1}\right|_{V_2=0} \\
&= Y_{a_{21}} = 0
\end{aligned}$$

Figure 5.20 b) shows the equivalent circuit for the calculation of $Y_{M_{12}}$ and $Y_{M_{22}}$. $Y_{M_{12}}$ is given by;

$$Y_{M_{12}} = \left.\frac{I'_1}{V_2}\right|_{V_1=0}$$

$$
\begin{aligned}
&= \left.\frac{I_1}{V_2}\right|_{V_2=0} \\
&= Y_{a_{21}}
\end{aligned}
$$

$Y_{M_{22}}$ is given by;

$$
\begin{aligned}
Y_{M_{22}} &= \left.\frac{I_2'}{V_2}\right|_{V_1=0} \\
&= \left.\frac{I_2 + I_2''}{V_2}\right|_{V_2=0} \\
&= Y_{a_{22}} + Y_2
\end{aligned}
$$

Since $Y_1 = Y_f(1 - A_v)$ and $Y_2 = Y_f - \frac{Y_f}{A_v}$ we can write $\left[\boldsymbol{Y_M'}\right]$ as indicated below:

$$
\left[\boldsymbol{Y_M'}\right] = \begin{bmatrix} Y_{a_{11}} + Y_f(1 - A_v) & 0 \\ \\ Y_{a_{21}} & Y_{a_{22}} + Y_f - \frac{Y_f}{A_v} \end{bmatrix} \tag{5.28}
$$

Solution of problem 5.10

The chain parameters can be defined by the set of eqns indicated below:

$$\begin{cases} V_1 = A_{11}\,V_2 - A_{12}\,I_2 \\ \\ I_1 = A_{21}\,V_2 - A_{22}\,I_2 \end{cases} \tag{5.29}$$

while the Y parameters can be defined by the following set of eqns:

$$\begin{cases} I_1 = Y_{11}\,V_1 + Y_{12}\,V_2 \\ \\ I_2 = Y_{21}\,V_1 + Y_{22}\,V_2 \end{cases} \tag{5.30}$$

Solving the first eqn of 5.29 in order to obtain I_2 we have:

$$I_2 = \frac{-1}{A_{12}}\,V_1 + \frac{A_{11}}{A_{12}}\,V_2 \tag{5.31}$$

Using the expression for I_2, given by this last eqn, on the second eqn of 5.29 we obtain:

$$\begin{aligned} I_1 &= A_{21}\,V_2 - A_{22}\left(\frac{-1}{A_{12}}\,V_1 + \frac{A_{11}}{A_{12}}\,V_2\right) \\ &= \frac{A_{22}}{A_{12}}\,V_1 + \left(A_{21} - \frac{A_{22}\,A_{11}}{A_{12}}\right)V_2 \end{aligned} \tag{5.32}$$

Comparing eqns 5.31 and 5.32 with the set of eqns defined by 5.30 we can write:

$$\begin{aligned} Y_{11} &= \frac{A_{22}}{A_{12}} \\ Y_{12} &= \left(A_{21} - \frac{A_{22}\,A_{11}}{A_{12}}\right) \\ Y_{21} &= \frac{-1}{A_{12}} \\ Y_{22} &= \frac{A_{11}}{A_{12}} \end{aligned}$$

Solution of problem 5.11

- *Circuit a):*

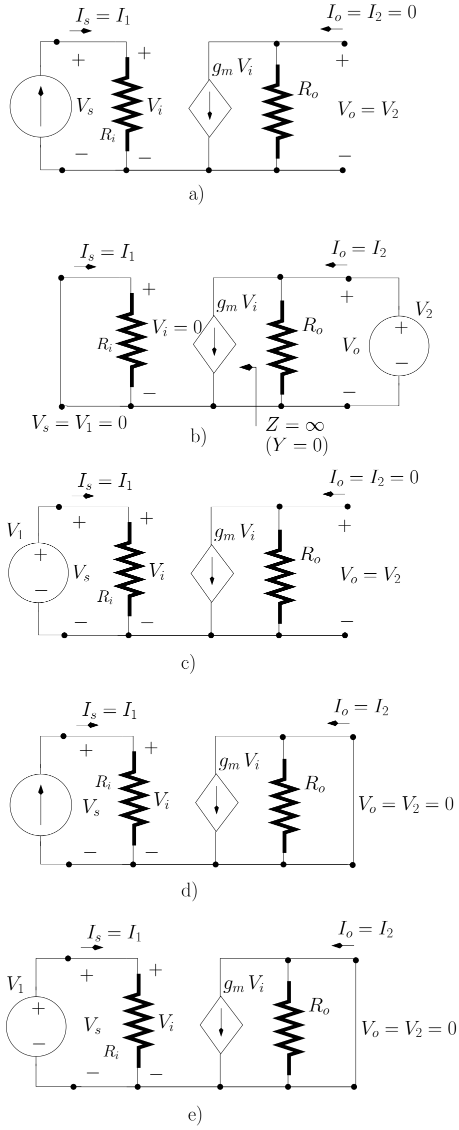


Figure 5.21: *Calculation of: a) Z_{11} and A_{21} ; b) Y_{22}; c) A_{11}; d) A_{22}; e) A_{12}.*

1. The input impedance ($I_o = 0$) corresponds to Z_{11}. Figure 5.21 a) shows the equivalent circuit for the calculation of Z_{11}.

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= R_i \\ &= 2.5 \ \text{k}\Omega \end{aligned}$$

2. The output impedance ($V_s = 0$) corresponds to $(Y_{22})^{-1}$. Figure 5.21 b) shows the equivalent circuit for the calculation of Y_{22}

$$\begin{aligned} Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= \frac{1}{R_o} \end{aligned}$$

Thus, the output impedance is $R_o = 10\ \text{k}\Omega$.

3. The voltage gain V_o/V_s ($I_o = 0$) corresponds to $(A_{11})^{-1}$. Figure 5.21 c) shows the equivalent circuit for the calculation of A_{11}

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \frac{V_1}{-g_m\, V_1\, R_o} \\ &= \frac{1}{-g_m\, R_o} \end{aligned}$$

Thus, the voltage gain is $-g_m\, R_o = -400$.

4. The current gain I_o/I_s ($V_o = 0$) corresponds to $-(A_{22})^{-1}$. Figure 5.21 d) shows the equivalent circuit for the calculation of A_{22}.

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{\frac{V_1}{R_i}}{-g_m\, V_1} \\ &= \frac{1}{-g_m\, R_i} \end{aligned}$$

Thus, the current gain is $g_m\, R_i = 100$.

5. The transimpedance gain V_o/I_s ($V_o = 0$) corresponds to $(A_{21})^{-1}$. Figure 5.21 a) shows the equivalent circuit for the calculation of A_{21}.

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{\frac{V_1}{R_i}}{-g_m\, V_1\, R_o} \\ &= \frac{1}{-g_m\, R_i\, R_o} \end{aligned}$$

The transimpedance gain is $-g_m\, R_i\, R_o = -1\ \text{M}\Omega$.

6. The transconductance gain I_o/V_s ($V_o = 0$) corresponds to $-(A_{12})^{-1}$. Figure 5.21 e) shows the equivalent circuit for the calculation of A_{12}.

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= \frac{V_1}{-g_m\, V_1} \\ &= \frac{1}{-g_m} \end{aligned}$$

The transconductance gain is $g_m = 40$ mS.

- *Circuit b):*

1. The input impedance ($I_o = 0$) corresponds to Z_{11}. Figure 5.22 a) shows the equivalent circuit for the calculation of Z_{11}. For this circuit we can write:

$$\begin{aligned} I_1 &= \frac{V_1}{R_i} - g_m\, V_i \\ &= V_1\, \frac{1 + g_m\, R_i}{R_i} \end{aligned}$$

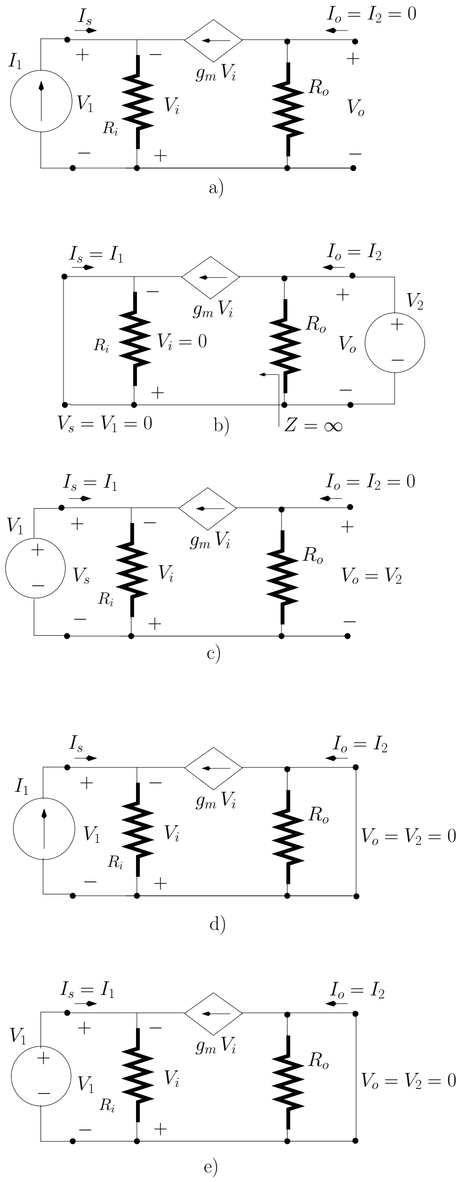


Figure 5.22: *Calculation of: a) Z_{11} and A_{21} ; b) Y_{22}; c) A_{11}; d) A_{22}; e) A_{12}.*

$$
\begin{aligned}
Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\
&= \frac{R_i}{1 + g_m R_i} \\
&= 24.8\ \Omega
\end{aligned}
$$

2. The output impedance ($V_s = 0$) corresponds to $(Y_{22})^{-1}$. Figure 5.22 b) shows the equivalent circuit for the calculation of Y_{22}.

$$
\begin{aligned}
Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\
&= \frac{1}{R_o}
\end{aligned}
$$

Thus, the output impedance is $R_o = 10\ \mathrm{k}\Omega$.

3. The voltage gain V_o/V_s ($I_o = 0$) corresponds to $(A_{11})^{-1}$. Figure 5.22 c) shows the

equivalent circuit for the calculation of A_{11}.

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \frac{-V_i}{-g_m\,V_i\,R_o} \\ &= \frac{1}{g_m\,R_o} \end{aligned}$$

Thus, the voltage gain is $g_m\,R_o = 400$.

4. The current gain I_o/I_s ($V_o = 0$) corresponds to $-(A_{22})^{-1}$. Figure 5.22 d) shows the equivalent circuit for the calculation of A_{22}.

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{\frac{V_1}{R_i} - g_m\,V_i}{-g_m\,V_i} \\ &= \frac{V_1\,\frac{1+g_m\,R_i}{R_i}}{g_m\,V_1} \\ &= \frac{1+g_m\,R_i}{g_m\,R_i} \end{aligned}$$

Thus, the current gain is -0.99.

5. The transimpedance gain V_o/I_s ($V_o = 0$) corresponds to $(A_{21})^{-1}$. Figure 5.22 a) shows the equivalent circuit for the calculation of A_{21}.

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{V_1\,\frac{1+g_m\,R_i}{R_i}}{g_m\,V_1\,R_o} \\ &= \frac{1+g_m\,R_i}{g_m\,R_i\,R_o} \end{aligned}$$

The transimpedance gain is 9.9 kΩ.

6. The transconductance gain I_o/V_s ($V_o = 0$) corresponds to $-(A_{12})^{-1}$. Figure 5.22 e) shows the equivalent circuit for the calculation of A_{12}.

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= \frac{V_1}{g_m\,V_1} \\ &= \frac{1}{g_m} \end{aligned}$$

The transconductance gain is $-g_m = -40$ mS.

- *Circuit c):*

1. The input impedance ($I_o = 0$) corresponds to Z_{11}. Figure 5.23 a) shows the equivalent circuit for the calculation of Z_{11}. For this circuit we can write:

$$\begin{aligned} I_1 &= \frac{V_i}{R_i} \\ V_2 &= R_o \times \left(g_m\,V_i + \frac{V_i}{R_i}\right) \\ V_1 &= V_i + V_o \end{aligned}$$

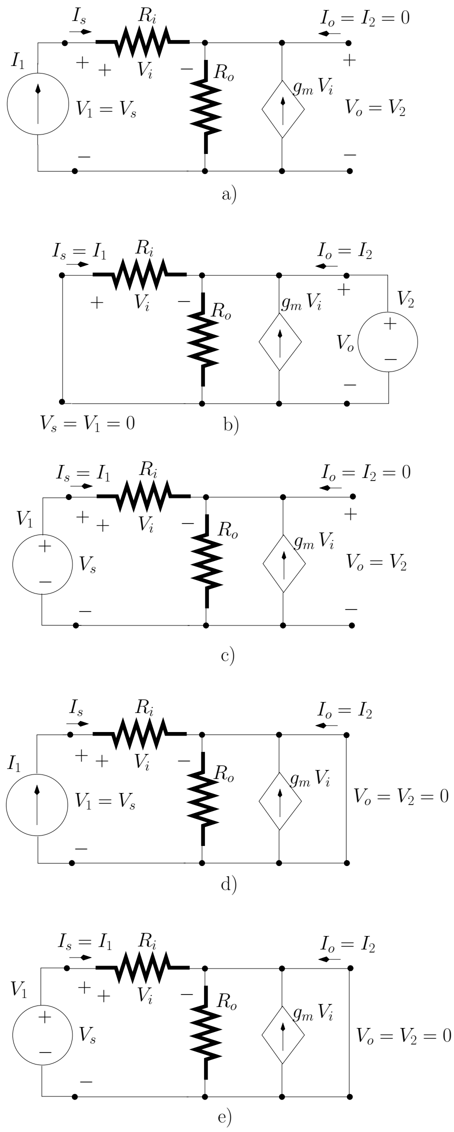


Figure 5.23: *Calculation of: a) Z_{11} and A_{21} ; b) Y_{22}; c) A_{11}; d) A_{22}; e) A_{12}.*

and

$$\begin{aligned} Z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ &= R_i + R_o\,(1 + g_m\,R_i) \\ &= 1.013\ \text{M}\Omega \end{aligned}$$

2. The output impedance ($V_s = 0$) corresponds to $(Y_{22})^{-1}$. Figure 5.23 b) shows the equivalent circuit for the calculation of Y_{22}.

$$\begin{aligned} Y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0} \\ &= \frac{g_m\,V_2 + \frac{V_2}{R_o} + \frac{V_2}{R_i}}{V_2} \\ &= \frac{g_m\,R_o\,R_i + R_o + R_i}{R_o\,R_i} \end{aligned}$$

Thus, the output impedance is $24.7\ \Omega$.

3. The voltage gain V_o/V_s ($I_o = 0$) corresponds to $(A_{11})^{-1}$. Figure 5.23 c) shows the equivalent circuit for the calculation of A_{11}.

$$\begin{aligned} A_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} \\ &= \frac{V_i + V_o}{V_o} \\ &= \frac{V_i + R_o \times \left(g_m\, V_i + \frac{V_i}{R_i}\right)}{R_o \times \left(g_m\, V_i + \frac{V_i}{R_i}\right)} \\ &= \frac{R_i + R_o\,(1 + g_m\, R_i)}{R_o\,(1 + g_m\, R_i)} \end{aligned}$$

Thus, the voltage gain is 0.998.

4. The current gain I_o/I_s ($V_o = 0$) corresponds to $-(A_{22})^{-1}$. Figure 5.23 d) shows the equivalent circuit for the calculation of A_{22}.

$$\begin{aligned} A_{22} &= \left.\frac{I_1}{-I_2}\right|_{V_2=0} \\ &= \frac{\frac{V_i}{R_i}}{g_m\, V_i + \frac{V_i}{R_i}} \\ &= \frac{1}{1 + g_m\, R_i} \end{aligned}$$

Thus, the current gain is -101.

5. The transimpedance gain V_o/I_s ($V_o = 0$) corresponds to $(A_{21})^{-1}$. Figure 5.23 a) shows the equivalent circuit for the calculation of A_{21}.

$$\begin{aligned} A_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} \\ &= \frac{\frac{V_i}{R_i}}{R_o \times \left(g_m\, V_i + \frac{V_i}{R_i}\right)} \\ &= \frac{1}{R_o\,(1 + g_m\, R_i)} \end{aligned}$$

The transimpedance gain is 1.01 MΩ.

6. The transconductance gain I_o/V_s ($V_o = 0$) corresponds to $-(A_{12})^{-1}$. Figure 5.23 e) shows the equivalent circuit for the calculation of A_{12}.

$$\begin{aligned} A_{12} &= \left.\frac{V_1}{-I_2}\right|_{V_2=0} \\ &= \frac{V_i}{g_m\, V_i + \frac{V_i}{R_i}} \\ &= \frac{R_i}{g_m\, R_i + 1} \end{aligned}$$

The transconductance gain is -40.4 mS.

Solution of problem 5.12

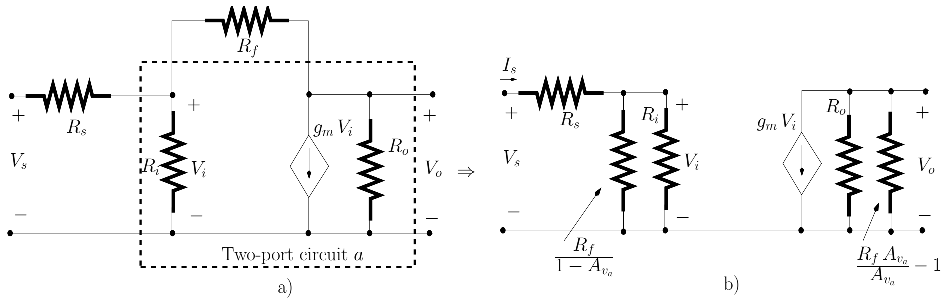


Figure 5.24: *a) Two-port network a) application of Miller's theorem.*

The two-port network a indicated in figure 5.24 a) (not including R_s and R_f) can be characterised by an admittance representation given by

$$\begin{aligned} Y_{11} &= \frac{1}{R_i} \\ &= 0.2 \text{ mS} \\ Y_{12} &= 0 \\ Y_{21} &= g_m \\ &= 40 \text{ mS} \\ Y_{22} &= \frac{1}{R_o} \\ &= 0.2 \text{ mS} \end{aligned}$$

and $Y_f = 1/R_f = 0.02$ mS. Since $|Y_{21}| >> |Y_f|$ and $|Y_{22}| >> |Y_f|$ we can write:

$$\begin{aligned} A_{v_a} &= \frac{V_o}{V_i} \\ &\simeq \frac{-Y_{21}}{Y_{22}} \\ &= -200 \end{aligned}$$

Figure 5.24 b) shows the circuit resulting from the application of Miller's theorem. The input impedance and the voltage gain V_o/V_s can be obtained assuming that a voltage source V_s is applied to the circuit. The input impedance R_{in} is given by:

$$\begin{aligned} R_{in} &= \frac{V_s}{I_s} \\ &= R_s + \left[\frac{R_f}{1 - A_{v_a}} \;||\; R_i\right] \\ &= 309.4 \ \Omega \end{aligned}$$

In order to calculate the voltage gain we can write

$$V_o = -g_m V_i \left[\frac{R_f A_{v_a}}{A_{v_a} - 1} \;||\; R_o\right]$$

and

$$V_i = V_s \frac{\left[\frac{R_f}{1-A_{v_a}} \;||\; R_i\right]}{\left[\frac{R_f}{1-A_{v_a}} \;||\; R_i\right] + R_s}$$

The voltage gain can be obtained as follows

$$\begin{aligned} A_V &= \frac{V_o}{V_s} = \frac{V_o}{V_i} \times \frac{V_i}{V_s} \\ &= -g_m \left[\frac{R_f\, A_{v_a}}{A_{v_a} - 1} \,||\, R_o \right] \times \frac{\left[\frac{R_f}{1 - A_{v_a}} \,||\, R_i \right]}{\left[\frac{R_f}{1 - A_{v_a}} \,||\, R_i \right] + R_s} \\ &= 122.4 \end{aligned}$$

Solution of problem 5.13

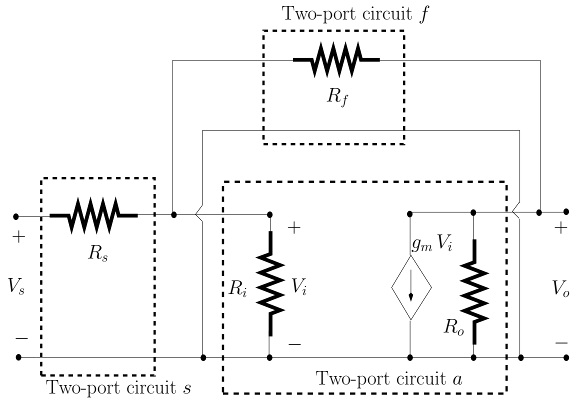


Figure 5.25: *Two-port network*

Figure 5.25 shows the circuit of the previous problem as an interconnection of two-ports. The two-port networks a and f can be characterised by admittance parameters as follows:

$$[\boldsymbol{Y_a}] = \begin{bmatrix} \frac{1}{R_i} & 0 \\ g_m & \frac{1}{R_o} \end{bmatrix} \tag{5.33}$$

$$[\boldsymbol{Y_f}] = \begin{bmatrix} \frac{1}{R_f} & \frac{-1}{R_f} \\ \frac{-1}{R_f} & \frac{1}{R_f} \end{bmatrix} \tag{5.34}$$

The parallel connection of the two-port network a with the two-port network f is given by the sum of $[\boldsymbol{Y_a}]$ with $[\boldsymbol{Y_f}]$ that is

$$[\boldsymbol{Y_{a+f}}] = \begin{bmatrix} \frac{1}{R_i} + \frac{1}{R_f} & -\frac{1}{R_f} \\ g_m - \frac{1}{R_f} & \frac{1}{R_o} + \frac{1}{R_f} \end{bmatrix} \tag{5.35}$$

Converting $[\boldsymbol{Y_{a+f}}]$ to a chain representation we get:

$$[\boldsymbol{A_{a+f}}] = \begin{bmatrix} \frac{R_f+R_o}{R_o\,(1-g_m\,R_f)} & \frac{R_f}{1-g_m\,R_f} \\ \frac{R_f+R_o+R_i(1+g_m\,R_o)}{R_i\,R_o\,(1-g_m\,R_f)} & \frac{R_f+R_i}{R_i\,(1-g_m\,R_f)} \end{bmatrix} \tag{5.36}$$

The two-port networks s can be characterised by a chain representation as follows:

$$[\boldsymbol{A_s}] = \begin{bmatrix} 1 & R_s \\ 0 & 1 \end{bmatrix} \tag{5.37}$$

The two-port network can be characterised by an equivalent chain representation given by:

$$[\boldsymbol{A_{eq}}] = [\boldsymbol{A_s}] \times [\boldsymbol{A_{a+f}}]$$

The voltage gain can now be obtained as follows:

$$\begin{aligned} A_V &= \frac{1}{A_{eq_{11}}} \\ &= -125.8 \end{aligned}$$

and the input impedance can be obtained (see table in appendix C) as follows:

$$\begin{aligned} R_{in} &= Z_{eq_{11}} = \frac{A_{eq_{11}}}{A_{eq_{21}}} \\ &= 329 \;\; \Omega \end{aligned}$$

Comparing these values with those obtained in the previous problem we conclude that the application of Miller's theorem introduces an error of about 6% in the calculation of the input impedance and an error of 2.8% in the calculation of the voltage gain. These errors are relatively small and it should be noticed that the application of Miller's theorem leads to simpler calculations.

Chapter 6

Basic electronic amplifier building blocks

Solution of problem 6.1

The gain of a non-inverter amplifier is

$$A_v = \frac{R_2}{R_1} + 1$$

For a gain of 20 we have that $R_2 = 19\, R_1$. Choosing $R_1 = 1$ kΩ we obtain $R_2 = 19$ kΩ.

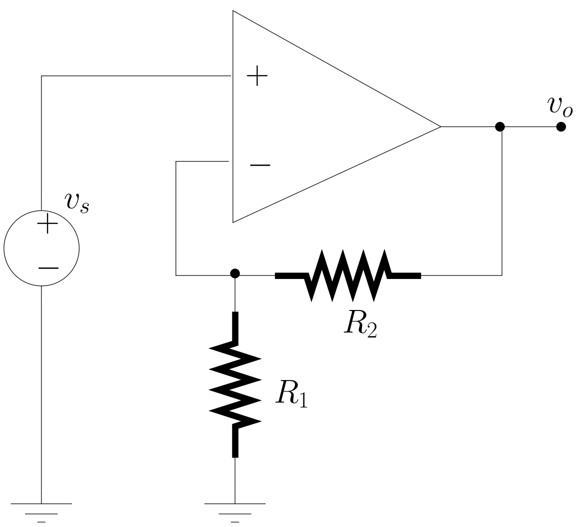


Figure 6.1: *Non-inverter amplifier.*

Solution of problem 6.2

The voltage gain is

$$\begin{aligned} A_v &= -\frac{R_2}{R_1} \\ &= -5 \end{aligned}$$

The input impedance is $R_1 = 1$ kΩ.

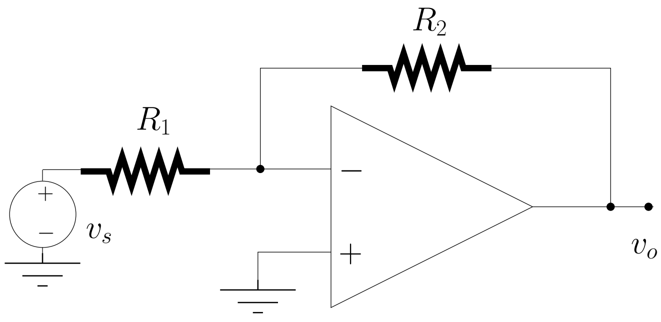


Figure 6.2: *Inverting amplifier.*

Solution of problem 6.3

This circuit can be implemented with an inverting amplifier with $R_2 = R_1$. We can choose $R_2 = R_1 = 1$ kΩ.

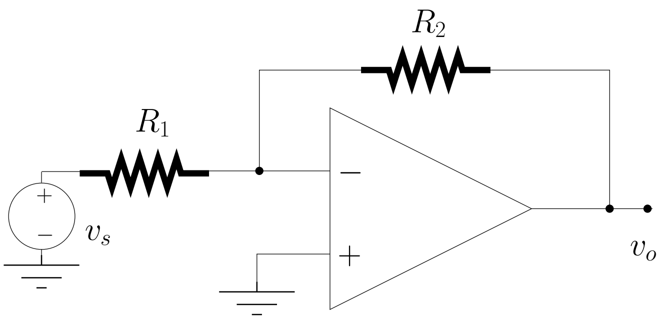


Figure 6.3: *Inverting amplifier.*

Solution of problem 6.4

Figure 6.4 shows the circuit designed with $R = 1$ kΩ. Note that $v_o' = -(v_{i1} + v_{i2} + v_{i3})$, since all resistances are equal. The second op-amp with two equal resistances, also chosen as $R = 1$ kΩ, implements an amplifier with a voltage gain of -1 such that $v_o = -v_o' = v_{i1} + v_{i2} + v_{i3}$.

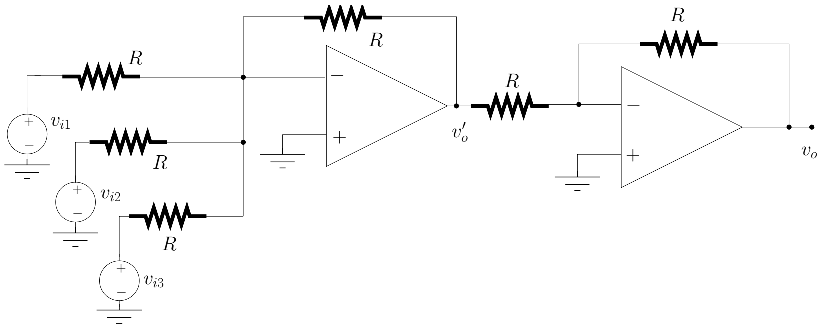


Figure 6.4: *Adder amplifier.*

Solution of problem 6.5

$$
\begin{aligned}
V_o &= \frac{R_2}{R_1} \times \frac{2\,R_3 + R_4}{R_4}\,(v_{sb} - v_{sa}) \\
&= 3\,(v_{sb} - v_{sa})
\end{aligned}
$$

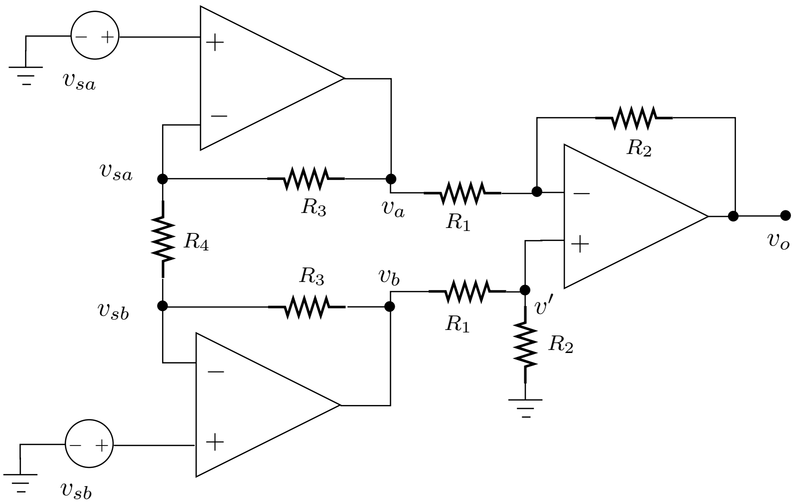


Figure 6.5: *Instrumentation amplifier.*

Solution of problem 6.6

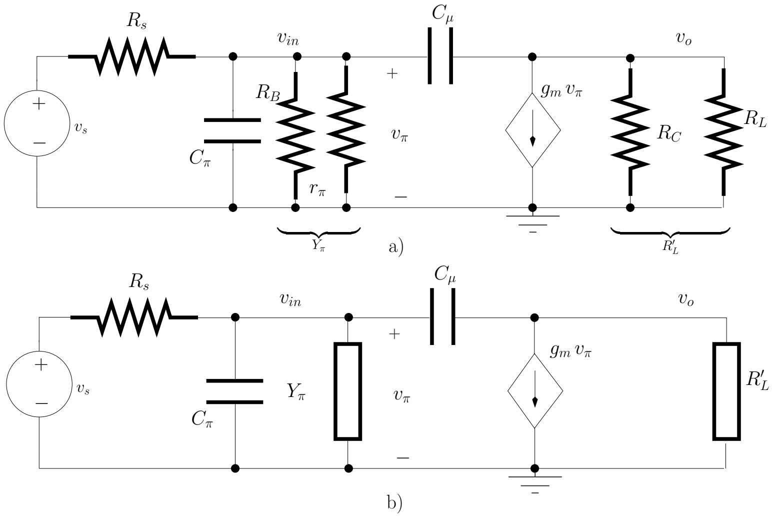


Figure 6.6: *a) Common-emitter small-signal equivalent amplifier for the mid and high-frequency ranges. b) Equivalent representation.*

Figure 6.6 b) shows the common-emitter small-signal equivalent amplifier for the mid and high-frequency ranges. Using the nodal analysis method we can write:

$$(v_s - v_\pi)\,Y_s = v_\pi(Y_\pi + j\,\omega\,C_\pi) + (v_\pi - v_o)j\,\omega\,C_\mu \tag{6.1}$$

$$(v_\pi - v_o)j\,\omega\,C_\mu = g_m\,v_\pi + \frac{v_o}{R'_L} \tag{6.2}$$

with

$$Y_\pi = \frac{1}{R_B} + \frac{1}{r_\pi}$$
$$Y_s = \frac{1}{R_s}$$
$$R'_L = R_L||R_C$$

Equation 6.1 can be rewritten as follows:

$$v_s\,Y_s = v_\pi(Y'_\pi + j\,\omega\,C_\pi) + (v_\pi - v_o)j\,\omega\,C_\mu \tag{6.3}$$

with

$$Y'_\pi = \frac{1}{R_s} + \frac{1}{R_B} + \frac{1}{r_\pi}$$

Solving the set of eqns given by 6.2 and 6.3 to obtain the voltage transfer function, v_o/v_s, we get:

$$A_v = \frac{Y_s\,R'_L\,(j\,\omega C_\mu - g_m)}{Y'_\pi(1 + j\,\omega\,R'_L C_\mu) + j\,\omega[C_\pi + C_\mu(1 + g_m R'_L)] - \omega^2\,C_\mu C_\pi R'_L}$$

Solution of problem 6.7

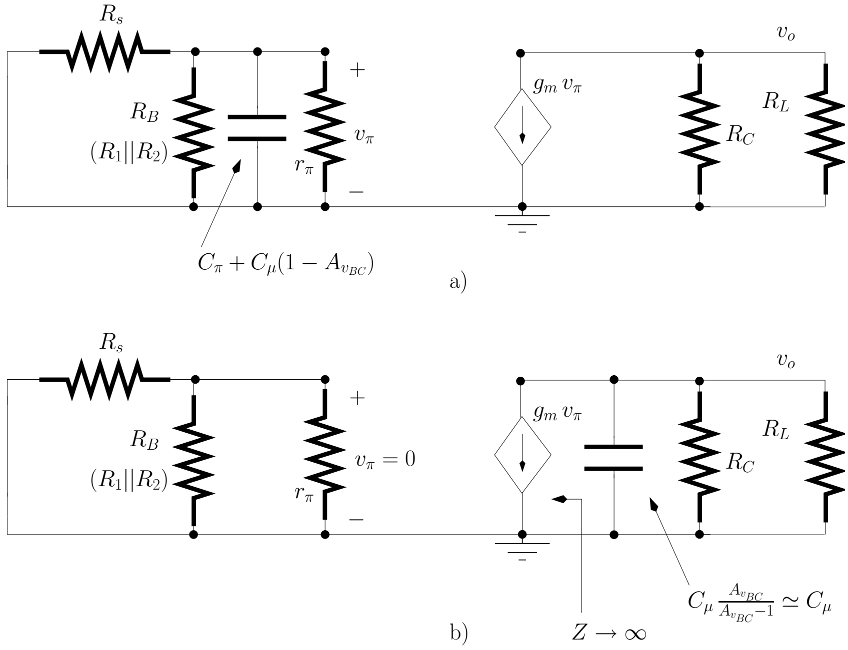


Figure 6.7: *Calculation of the high-frequency time constants.*

Figure 6.7 a) shows the equivalent circuit to calculate the equivalent resistance at the terminals of $C_\pi + C_\mu(1 + g_m\, R'_L)$. Note that the voltage source v_s is replaced by a short-circuit. From this circuit it is clear that this resistance is

$$R_{eq_1} = R_s||R_B||r_\pi$$

Figure 6.7 b) shows the equivalent circuit to calculate the equivalent resistance at the terminals of C_μ. Again the voltage source v_s is replaced by a short-circuit. Since $v_\pi = 0$ the voltage controlled current source is effectively an open-circuit. Hence, we have:

$$R_{eq_1} = R_L$$

Solution of problem 6.8

The current gain is given by:

$$\frac{i_c}{i_b} \simeq \frac{g_m\, r_\pi}{1 + j\,\omega\,(C_\mu + C_\pi)\, r_\pi}$$

$\omega_T = 2\,\pi\, f_T$ is the frequency for which the current gain is unity,

$$\left|\frac{i_c}{i_b}\right| = 1 \text{ (0 dB)}$$

that is:

$$\begin{aligned}
\frac{g_m\, r_\pi}{\sqrt{1 + (\omega_T\,(C_\mu + C_\pi)\, r_\pi)^2}} &= 1 \\
\Leftrightarrow\ 1 + r_\pi^2\,(C_\mu + C_\pi)^2\,\omega_T^2 &= (g_m\, r_\pi)^2 \\
\Leftrightarrow\ \omega_T^2 &= \frac{(g_m\, r_\pi)^2 - 1}{r_\pi^2\,(C_\mu + C_\pi)^2}
\end{aligned}$$

Since $g_m\, r_\pi >> 1$ we can write:

$$\begin{aligned}
\omega_T &\simeq \frac{g_m\, r_\pi}{r_\pi\,(C_\mu + C_\pi)} \\
&= \frac{g_m}{C_\mu + C_\pi}
\end{aligned}$$

Solution of problem 6.9

Figure 6.8 shows the equivalent circuit for the calculation of the f_T for FETs. For this circuit we can

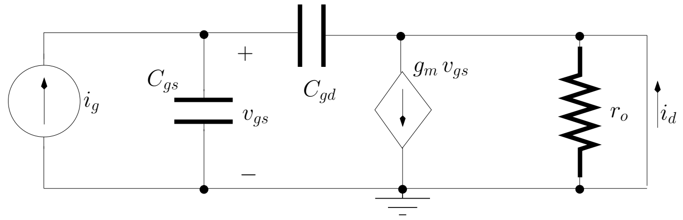


Figure 6.8: *Calculation of the f_T for FETs.*

write:

$$\begin{aligned} i_g &= j\,\omega\,C_{gs}\,v_{gs} + j\,\omega\,C_{gd}\,v_{gs} \\ i_d &= g_m\,v_{gs} - j\,\omega\,C_{gd}\,v_{gs} \end{aligned}$$

Note that the voltage across C_{gd} is v_{gs}. This is because the drain is short-circuited to the source.

Solving this set of eqns we obtain:

$$\frac{i_d}{i_d} = \frac{g_m - j\,\omega\,C_{gd}}{j\,\omega\,(C_{gd} + C_{gs})}$$

For the range of frequencies for which this model is valid we have that $g_m >> \omega\,C_{gd}$. Hence, we can write:

$$\frac{i_d}{i_d} \simeq \frac{g_m}{j\,\omega\,(C_{gd} + C_{gs})}$$

and the frequency $\omega_T = 2\,\pi\,f_T$ for which we have:

$$\left|\frac{i_d}{i_d}\right| = 1$$

is

$$\omega_T = \frac{g_m}{C_{gd} + C_{gs}}$$

Solution of problem 6.10

- *DC analysis:* Figure 6.9 shows the equivalent circuit for DC analysis. Since the gate current

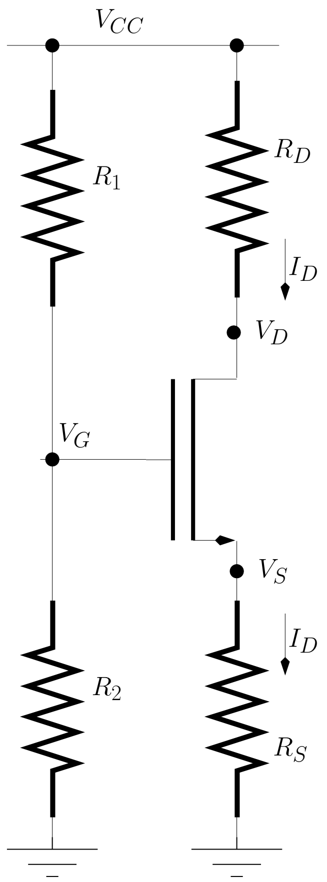


Figure 6.9: *Circuit for DC analysis.*

is zero we can write:

$$\begin{aligned} V_G &= V_{CC}\,\frac{R_2}{R_2+R_1} \\ &= 2\ \text{V} \end{aligned}$$

Assuming that the transistor operates in the saturation region, the drain (and the source) current is given by

$$I_D = \frac{1}{2}\,k_n\,\frac{W}{L}\,\left(V_{GS}-V_{Th}\right)^2 \tag{6.4}$$

and V_{GS} can be written as

$$\begin{aligned} V_{GS} &= V_G - V_S \\ &= V_G - R_S\,I_D \end{aligned}$$

Now eqn 6.4 can be written as

$$I_D = \frac{1}{2}\,k_n\,\frac{W}{L}\,\left(V_G - R_S\,I_D - V_{Th}\right)^2 \tag{6.5}$$

Solving we obtain

$$\left\{ \begin{array}{c} I_D = 1.5\ \text{mA} \quad \text{and} \quad V_{GS} = 1.85\ \text{V} \\ \\ \text{or} \\ \\ I_D = 68.5\ \text{mA} \quad \text{and} \quad V_{GS} = -4.9\ \text{V} \end{array} \right. \tag{6.6}$$

Clearly, the second pair of solutions is not valid since $V_{GS} < V_{Th}$. Hence we have $I_D = 1.5$ mA and $V_{GS} = 1.85$ V. Note that $V_{DS} = 5.8$ V is greater than $V_{GS} - V_{Th} = 0.85$ V. Thus, the FET operates in the saturation region as assumed previously.

- *Mid frequency range AC analysis:* Figure 6.10 shows the equivalent circuit for AC analysis.

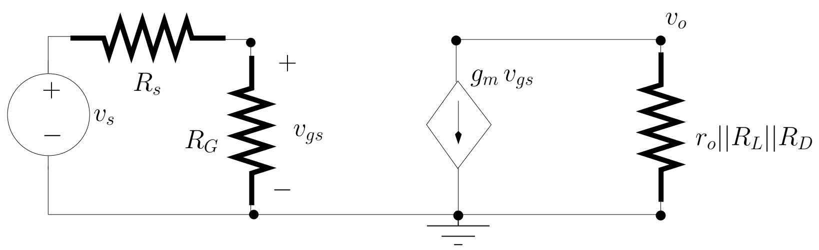


Figure 6.10: *Circuit for AC analysis.*

We can write the voltage gain as

$$\begin{aligned} A_v &= \frac{v_o}{v_s} \\ &= \frac{v_o}{v_{gs}} \times \frac{v_{gs}}{v_s} \\ &= -g_m\,(r_o||R_L||R_D)\;\frac{R_G}{R_G + R_s} \end{aligned}$$

with $R_G = R_1||R_2 = 16$ kΩ. g_m and r_o are given by

$$\begin{aligned} g_m &= \frac{2\,I_D}{V_{GS} - V_{Th}} \\ &= 3.5 \ \text{mA/V} \\ r_o &= \frac{V_A}{I_D} \\ &= 53 \ \text{k}\Omega \end{aligned}$$

Hence, the voltage gain is calculated to be -7.7.

- *Low-frequency cut-off frequency:* Figure 6.11 shows the equivalent circuit for the calculation of the time constants associated with each DC blocking (AC coupling) and by-pass capacitor using the short-circuit time constants method. Figure 6.11 a) shows the equivalent circuit for the calculation of the time constant associated with C_G where we replace C_G by a test voltage source in order to calculate equivalent resistance, R_{eq_G}, seen by this capacitor. Note that all remaining capacitors are replaced by short-circuits and the voltage source is replaced by a short-circuit. From this figure it clear that R_{eq_G} is

$$\begin{aligned} R_{eq_G} &= R_s + R_G \\ &= 16.1 \ \text{k}\Omega \end{aligned}$$

Figure 6.11 b) shows the equivalent circuit for the calculation of the time constant associated with C_S. Since there is no current flowing through R_s and R_G we have $v_{gs} = -v_t$ and we can write:

$$\frac{v_o}{R'_L} + \frac{v_o - v_t}{r_o} - g_m\,v_t = 0 \tag{6.7}$$

$$\frac{v_o - v_t}{r_o} - g_m\,v_t = \frac{v_t}{R_S} + i_t \tag{6.8}$$

with $R'_L = R_D||R_L$. Solving these eqns in order to obtain v_t/it we get:

$$R_{eq_S} = \frac{v_t}{i_t}$$

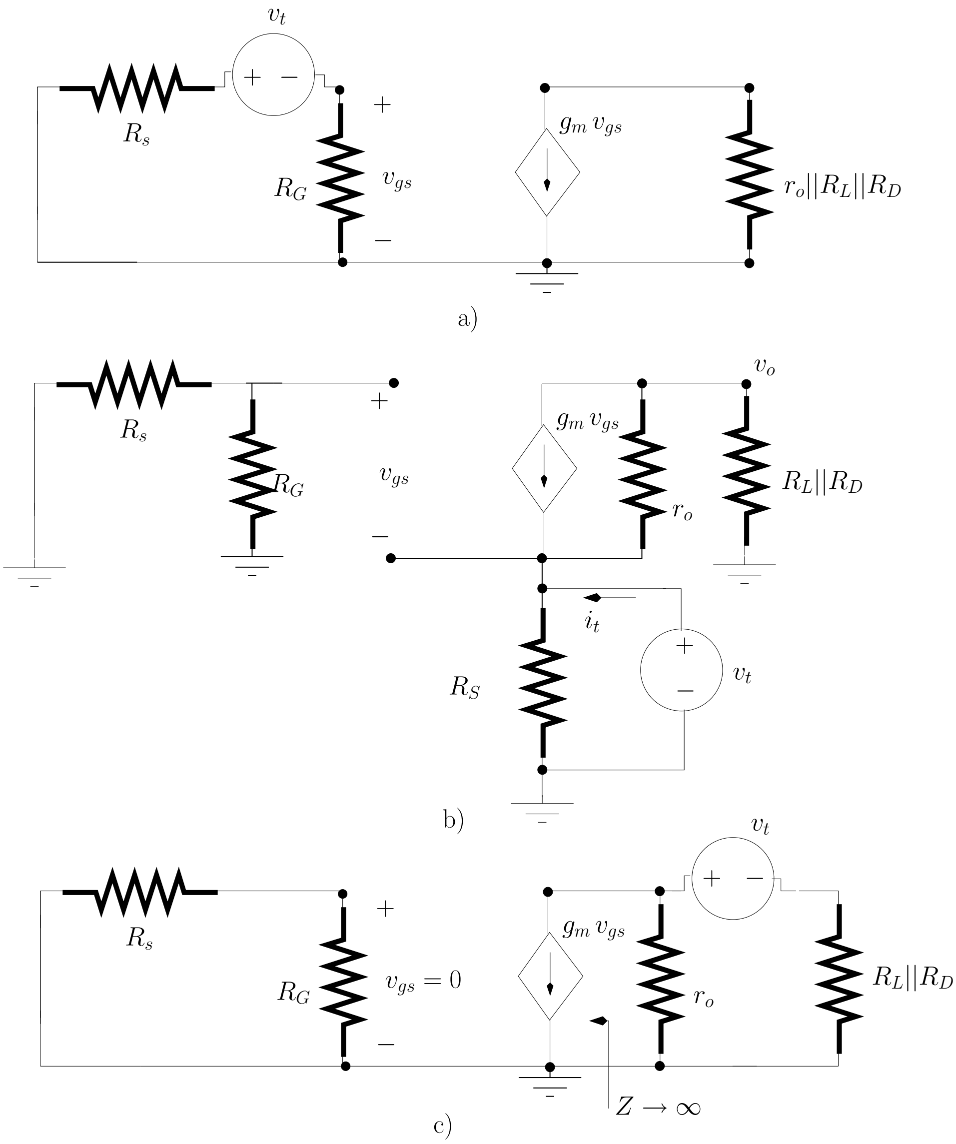


Figure 6.11: *Calculation of time constants.*

$$
\begin{aligned}
&= \frac{R_S\,(r_o + R_L')}{R_S(g_m\, r_o + 1) + r_o + R_L'} \\
&= 74.6\ \Omega
\end{aligned}
$$

Figure 6.11 c) shows the equivalent circuit for the calculation of the time constant associated with C_L. Since $v_{gs} = 0$ and we can write:

$$
\begin{aligned}
R_{eq_L} &= \frac{v_t}{i_t} \\
&= r_o||R_L||R_D \\
&= 2.2\ \text{k}\Omega
\end{aligned}
$$

and f_L is given by:

$$
\begin{aligned}
f_L &= \frac{1}{2\,\pi}\left(\frac{1}{R_{eq_G}\, C_G} + \frac{1}{R_{eq_S}\, C_S} + \frac{1}{R_{eq_L}\, C_L}\right) \\
&= 287.7\ \text{Hz}
\end{aligned}
$$

- *High-frequency cut-off frequency:* Figure 6.12 a) shows the equivalent circuit for the high-frequency range. By applying Miller's theorem to C_{gd} we obtain the circuit of figure 6.12 b)

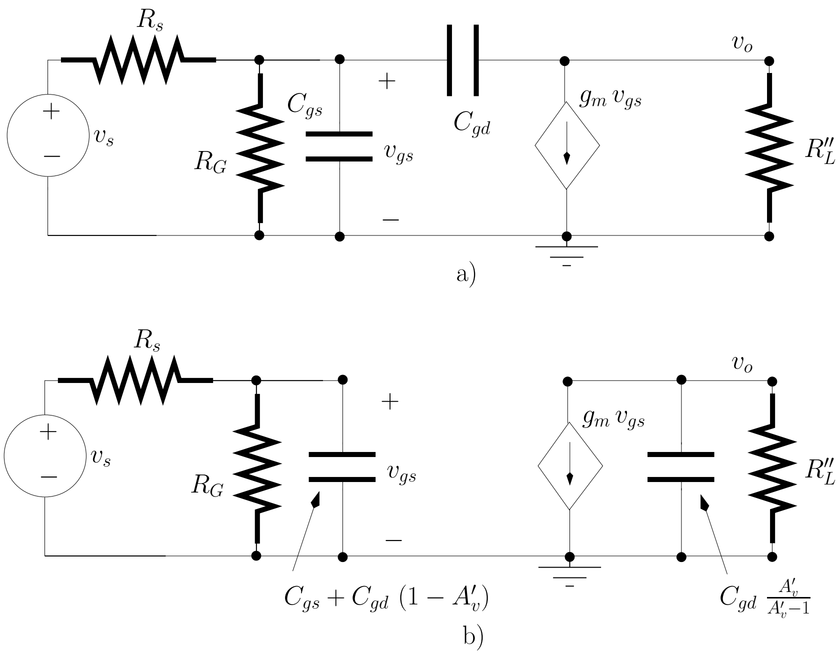


Figure 6.12: *a) Equivalent circuit for the high-frequency range. b) Application of Miller's theorem to C_{gd}.*

where A_v' is the gain between the gate and the drain;

$$\begin{aligned} A_v' &= \frac{v_o}{v_{gs}} \\ &\simeq -g_m\,R_L'' \\ &= -7.7 \end{aligned}$$

with $R_L'' = R_L' \,||\, r_o$. The two time constants are:

$$\begin{aligned} \tau_1 &= (R_G||R_s)\,[C_{gs} + C_{gd}\,(1+g_m\,R_L'')] \\ &= 3.7 \text{ ns} \\ \tau_2 &= R_L''\,C_{gd}\,\frac{g_m\,R_L''}{1+g_m\,R_L''} \\ &= 3.9 \text{ ns} \end{aligned}$$

The cut-off frequency is

$$\begin{aligned} f_H &= \frac{1}{2\,\pi\,(\tau_1+\tau_2)} \\ &= 20.9 \text{ MHz} \end{aligned}$$

Hence, the bandwidth is $f_H - f_L \simeq f_H = 20.9$ MHz.

Solution of problem 6.11

- *DC analysis:* Figure 6.13 shows the equivalent circuit for DC analysis. Neglecting the base

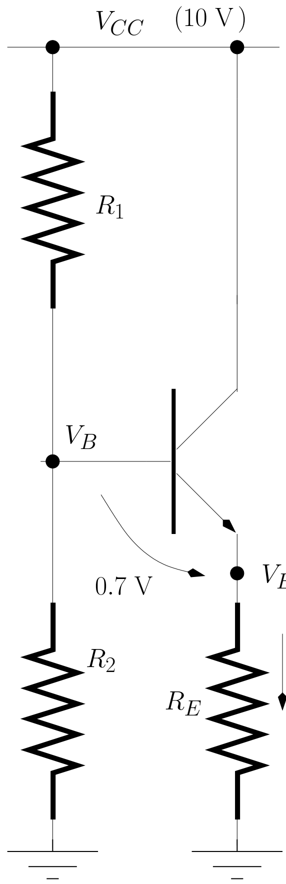


Figure 6.13: *Equivalent circuit for DC analysis.*

current ($I_B \simeq 0$) we can write:

$$\begin{aligned} V_B &= V_{CC}\frac{R_2}{R_2+R_1} \\ &= 5 \text{ V} \end{aligned}$$

and

$$\begin{aligned} V_E &= V_B - 0.7 \\ &= 4.3 \text{ V} \end{aligned}$$

The emitter current is

$$\begin{aligned} I_E &= \frac{V_E}{R_E} \\ &= 1 \text{ mA} \end{aligned}$$

The collector current is approximately equal to I_E.

- *AC analysis (mid-range):* The transistor transconductance is

$$\begin{aligned} g_m &= \frac{I_C}{V_T} \quad (V_T \simeq 25 \text{ mV}) \\ &= 40 \text{ mS} \end{aligned}$$

and r_π is

$$\begin{aligned} r_\pi &= \frac{\beta}{g_m} \\ &= 5\ \text{k}\Omega \end{aligned}$$

Figure 6.14 shows the equivalent circuit for the mid-frequency range AC analysis. The current

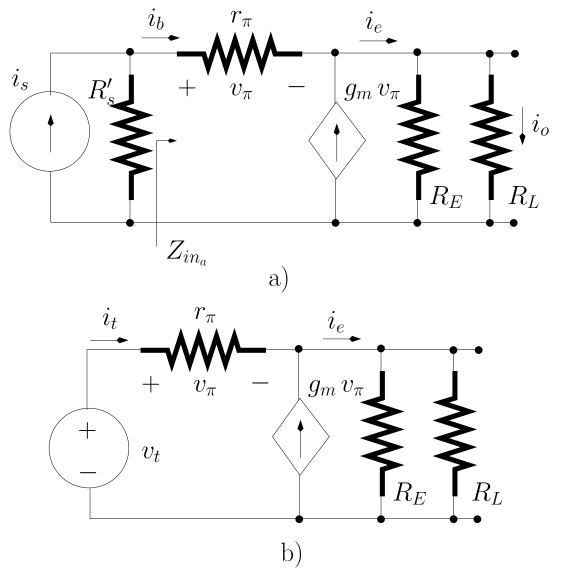


Figure 6.14: *a) Equivalent circuit for mid-frequency range AC analysis. b) Equivalent circuit for the calculation of Z_{in_a}.*

gain is given by

$$\begin{aligned} A_i &= \frac{i_o}{i_s} \\ &= \frac{i_o}{i_e} \times \frac{i_e}{i_b} \times \frac{i_b}{i_s} \end{aligned}$$

the partial gains i_o/i_e and i_b/i_s can be obtained from the resistive current divider expression while $i_e = (\beta+1)\, i_b$. Thus, we can write:

$$A_i = \frac{R_E}{R_E + R_L} \times (\beta+1) \times \frac{R'_s}{R'_s + Z_{in_a}} \tag{6.9}$$

where $R'_s = R_s || R_1 || R_2 = 5\ \text{k}\Omega$. Z_{in_a} can be calculated by applying first a test voltage, v_t, to the relevant part of the circuit as illustrated in figure 6.14 b). From this circuit we can write.

$$i_t = \frac{v_\pi}{r_\pi}$$

and

$$\begin{aligned} v_t &= v_\pi + i_e\, (R_E || R_L) \\ &= v_\pi + \left(\frac{v_\pi}{r_\pi} + g_m\, v_\pi \right) (R_E || R_L) \end{aligned}$$

Thus we have

$$\begin{aligned} Z_{in_a} &= \frac{v_t}{i_t} \\ &= r_\pi + (g_m\, r_\pi + 1)\, (R_E || R_L) \\ &= r_\pi + (\beta + 1)\, (R_E || R_L) \\ &= 676.7\ \text{k}\Omega \end{aligned}$$

Now the current gain can be calculated using eqn 6.9, that is, $A_i = 0.33$.

- *AC analysis (low-frequency range):* Figure 6.15 a) shows the equivalent circuit for the calcu-

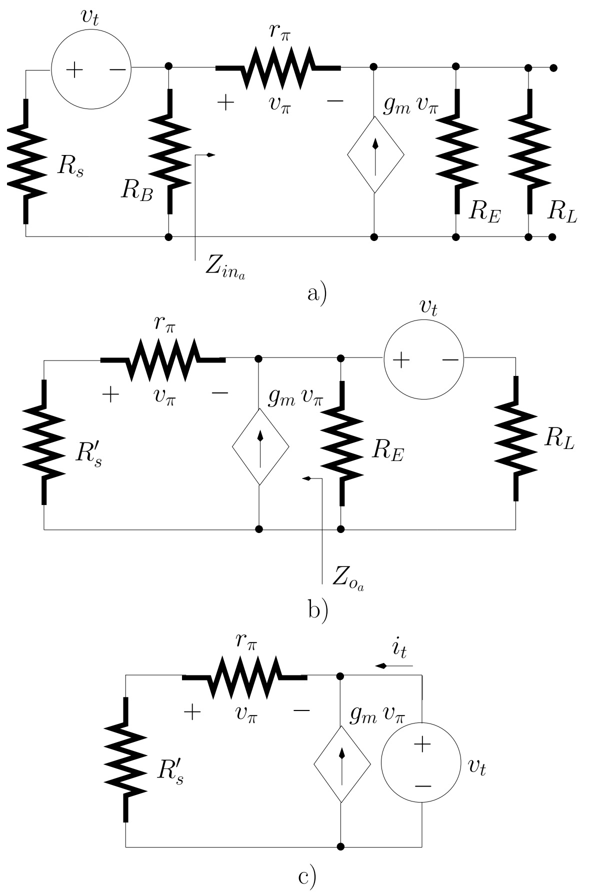


Figure 6.15: *Calculation of time constants. Low-frequency range.*

lation of the equivalent resistance seen by C_B where C_B is replaced by a test voltage v_t. From this circuit we can write

$$\begin{aligned} R_{eq_B} &= R_s + (R_B||Z_{in_a}) \\ &= 19.9\text{k}\Omega \end{aligned}$$

with $R_B = R_1||R_2$. Figure 6.15 b) shows the equivalent circuit for the calculation of the equivalent resistance seen by C_E;

$$R_{eq_E} = R_L + (R_E||Z_{o_a})$$

where Z_{o_a} can be obtained from the circuit of figure 6.15 c). We can write:

$$v_\pi = -v_t\,\frac{r_\pi}{r_\pi + R'_s}$$

and

$$\begin{aligned} i_t &= -g_m\,v_\pi + \frac{v_t}{r_\pi + R'_s} \\ &= g_m\,v_t\,\frac{r_\pi}{r_\pi + R'_s} + \frac{v_t}{r_\pi + R'_s} \end{aligned}$$

Now Z_{o_a} can be obtained as follows:

$$Z_{o_a} = \frac{v_t}{i_t}$$

$$
\begin{aligned}
&= \frac{r_\pi + R'_s}{g_m\, r_\pi + 1} \\
&= \frac{r_\pi + R'_s}{\beta + 1} \\
&= 49.8\ \Omega
\end{aligned}
$$

and R_{eq_E} is approximately equal to $R_L = 15$ kΩ. The cut-off frequency is:

$$
\begin{aligned}
f_L &= \frac{1}{2\,\pi}\left(\frac{1}{R_{eq_E}\,C_E} + \frac{1}{R_{eq_B}\,C_B}\right) \\
&= 2.7 \text{ Hz}
\end{aligned}
$$

- *AC analysis (high-frequency range):* Figure 6.16 a) shows the equivalent circuit at the high-

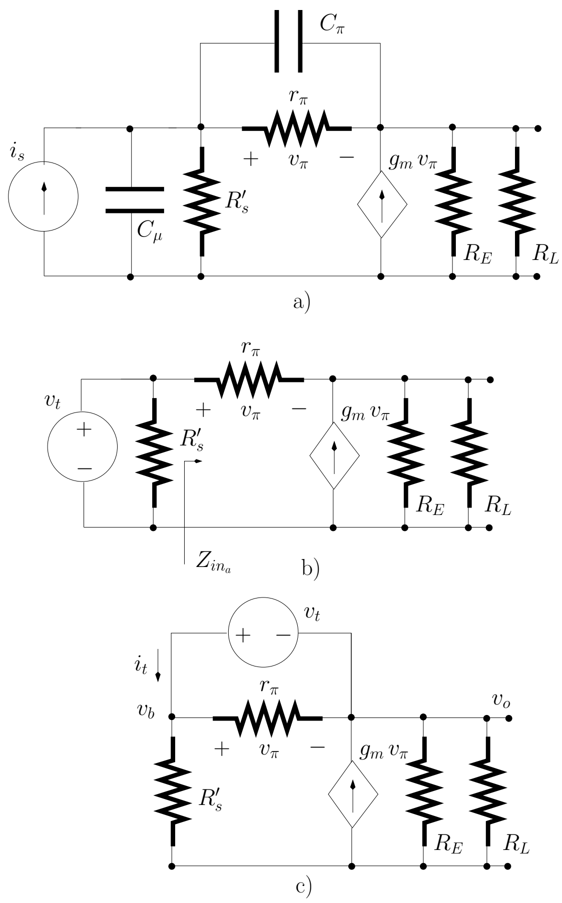


Figure 6.16: *a) High-frequency equivalent circuit b) Calculation of time constant associated with C_π. c) Calculation of time constant associated with C_μ.*

frequency regime. Figure 6.16 b) shows the equivalent circuit for the calculation of the resistance seen by C_μ where, according to the open-circuit time-constants method, i_s is replaced by an open-circuit and C_π is also replaced by an open-circuit. R_{eq_μ} is:

$$
\begin{aligned}
R_{eq_\mu} &= R'_s || Z_{in_a} \\
&= 5.0\ \text{k}\Omega
\end{aligned}
$$

Figure 6.16 c) shows the equivalent circuit for the calculation of the resistance seen by C_π. Again, i_s is replaced by an open-circuit and C_π is also replaced by an open-circuit. For this circuit we can write:

$$
i_t = \frac{v_t}{r_\pi} + \frac{v_b}{R'_s}
$$

But v_b is

$$v_b = v_t + \left(\frac{v_t}{r_\pi} + g_m\, v_t\right)\, R'_L$$

Hence we have:

$$i_t = \frac{v_t}{r_\pi} + \frac{v_t}{R'_s} + v_t\, \left(\frac{1 + g_m\, r_\pi}{r_\pi}\right)\, \frac{R'_L}{R'_s}$$

Finally,

$$\begin{aligned} R_{eq_\pi} &= \frac{v_t}{i_t} \\ &= r_\pi\, ||R'_s\, ||\, \frac{r_\pi\, R'_s}{(1 + g_m\, r_\pi)\, R'_L} \\ &= 36.9\ \ \Omega \end{aligned}$$

The cut-off frequency is:

$$\begin{aligned} f_H &= \frac{1}{2\,\pi}\, \left(\frac{1}{R_{eq_\mu}\, C_\mu + R_{eq_\pi}\, C_\pi}\right) \\ &= 10.2\ \ \text{MHz} \end{aligned}$$

and the bandwidth is $f_H - f_L \simeq f_H$.

Solution of problem 6.12

$$\sqrt{I_{ds_1}} - \sqrt{I_{ds_2}} = \sqrt{\frac{1}{2}\,k_n\,\frac{W}{L}v_s} \tag{6.10}$$

and

$$I_{ds_2} = I_Q - I_{ds_1} \tag{6.11}$$

Substituting I_{ds_2}, given by the last eqn, in eqn 6.10 we can write:

$$\sqrt{I_{ds_1}} - \sqrt{I_Q - I_{ds_1}} = \sqrt{\frac{1}{2}\,k_n\,\frac{W}{L}\,v_s}$$

Squaring both parts of the last eqn we get:

$$I_Q - \frac{1}{2}\,k_n\,\frac{W}{L}\,v_s^2 = 2\,\sqrt{I_{ds_1}\,(I_Q - I_{ds_1})} \tag{6.12}$$

After squaring again we can write:

$$4\,I_{ds_1}^2 - 4\,I_{ds_1}\,I_Q + \left(I_Q - \frac{1}{2}\,k_n\,\frac{W}{L}\,v_s^2\right)^2 = 0 \tag{6.13}$$

solving to obtain I_{ds_1} we get[1]:

$$\begin{aligned} I_{ds_1} &= \frac{I_Q}{2} + \frac{1}{2}\sqrt{I_Q^2 - \left(I_Q - \frac{1}{2}\,k_n\,\frac{W}{L}\,v_s^2\right)^2} \\ &= \frac{I_Q}{2} + \sqrt{I_Q\,k_n\,\frac{W}{L}}\;\frac{v_s}{2}\;\sqrt{1 - \frac{v_s^2/4\;k_n\,\frac{W}{L}}{I_Q}} \end{aligned}$$

Using eqn 6.11 we have

$$I_{ds_2} = \frac{I_Q}{2} - \sqrt{I_Q\,k_n\,\frac{W}{L}}\;\frac{v_s}{2}\;\sqrt{1 - \frac{v_s^2/4\;k_n\,\frac{W}{L}}{I_Q}}$$

[1]Note that the other solution for I_{ds_1} is not valid since I_{ds_1} increases with the increase of v_s!

Solution of problem 6.13

Figure 6.17 shows the equivalent circuit for DC analysis. Assuming that all transistors operate in the

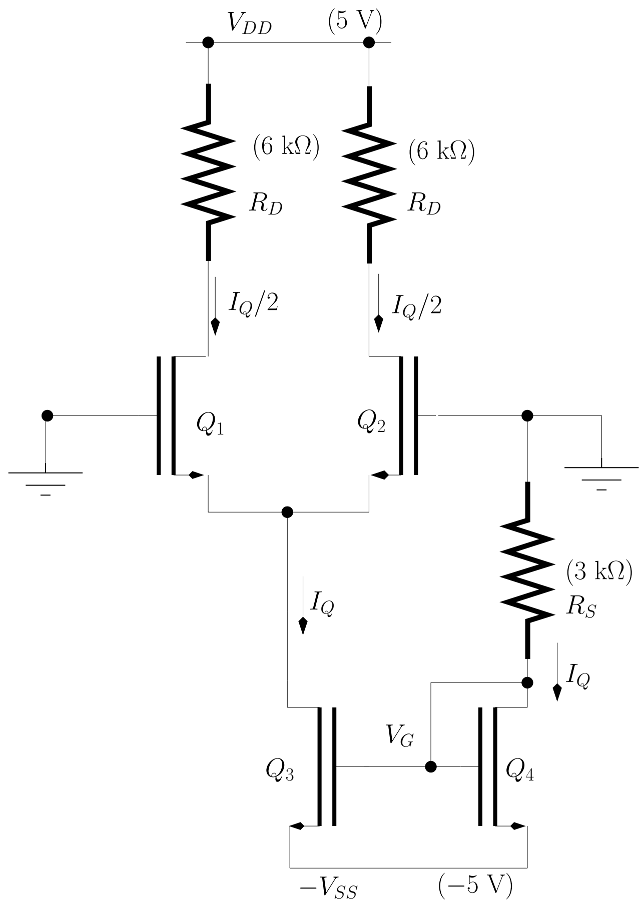


Figure 6.17: *Equivalent circuit for DC analysis.*

saturation region we can write:

$$\begin{aligned} I_Q &= \frac{1}{2}\,k_n\,\frac{W}{L}\,(V_{GS_4} - V_{Th})^2 \\ V_{GS_4} &= V_G - V_{SS} \\ V_G &= -I_Q\,R_S \end{aligned}$$

that is:

$$I_Q = \frac{1}{2}k_n\,\frac{W}{L}\,(-\frac{I_Q}{R_s} - V_{SS} - V_{Th})^2$$

Solving we obtain

$$\begin{cases} I_Q = 1 \text{ mA} \quad \text{and} \quad V_{GS_4} = 2 \text{ V} \\ \text{or} \\ I_Q = 1.8 \text{ mA} \quad \text{and} \quad V_{GS_4} = -0.4 \text{ V} \end{cases} \tag{6.14}$$

The second solution is not valid since $V_{GS_4} < V_{Th}$. The current that flows through Q_3 is I_Q and $V_{GS_3} = V_{GS_4}$. The current that biases Q_1 and Q_2 is $I_Q/2$ and $V_{GS_1} = V_{GS_2} = 1.71$ V. Note that all transistors operate in the saturation regime as assumed initially.

The voltage gain is

$$
\begin{aligned}
A_v &= \frac{g_m\,(R_D||r_o)}{2} \\
&= \frac{I_Q}{2(V_{GS_2}-V_{Th})}\left(R_D\;||\;\frac{2\,V_A}{I_Q}\right) \\
&= 4.1
\end{aligned}
$$

Solution of problem 6.14

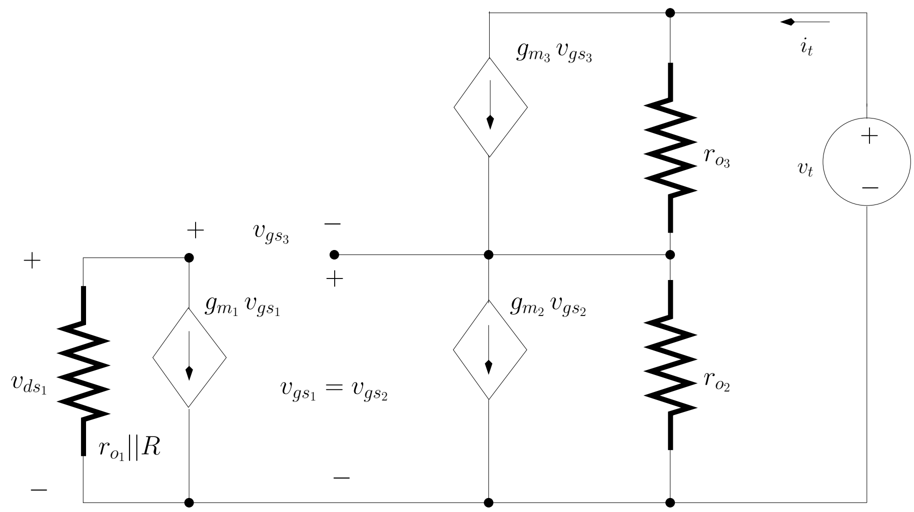


Figure 6.18: *Small-signal equivalent circuit for the current mirror.*

Figure 6.18 shows the small-signal equivalent circuit for the current mirror where we apply a test voltage source v_t. The output resistance is given by

$$R_o = \frac{v_t}{i_t}$$

For this circuit we can write:

$$i_t = g_{m_3}\,v_{gs_3} + \frac{v_t - v_{gs_2}}{r_{o_3}} \tag{6.15}$$

$$i_t = g_{m_2}\,v_{gs_2} + \frac{v_{gs_2}}{r_{o_2}} \tag{6.16}$$

$$\begin{aligned} v_{gs_3} &= v_{ds_1} - v_{gs_2} \\ &= -(r_{o_1}||R)\,g_{m_1} v_{gs_2} - v_{gs_2} \end{aligned} \tag{6.17}$$

Assuming that all three transistors are equal we have $r_{o_1} = r_{o_2} = r_{o_3} = r_o$ and $g_{m_1} = g_{m_2} = g_{m_3} = g_m$.

Solving the set of eqns 6.15–6.17 to obtain v_t/i_t we get:

$$R_o = r_o\,\frac{2\,(g_m\,r_o + 1) + g_m^2\,r_o\,(r_o||R)}{g_m\,r_o + 1}$$

Assuming that $g_m\,r_o >> 1$ and that $r_o << R$ we can write the last eqn as follows:

$$\begin{aligned} R_o &\simeq g_m\,r_o^2\,\underbrace{\frac{g_m\,r_o + 2}{g_m\,r_o}}_{\simeq 1} \\ &\simeq g_m\,r_o^2 \end{aligned}$$

Chapter 7

RF circuit analysis techniques

Solution of problem 7.1

$$
\begin{aligned}
\frac{V(k\Delta x)}{V([k-1]\Delta x)} &= \frac{\sqrt{\frac{L}{C} - \omega^2 \frac{L^2\Delta x^2}{4}} - j\omega\frac{L\Delta x}{2}}{\sqrt{\frac{L}{C} - \omega^2 \frac{L^2\Delta x^2}{4}} + j\omega\frac{L\Delta x}{2}} \\
&= \frac{\sqrt{\left(\frac{L}{C} - \frac{\omega^2 L^2 \Delta x^2}{2}\right)^2 + \omega^2 L^2 \Delta x^2 \left(\frac{L}{C} - \frac{\omega^2 L^2\Delta x^2}{4}\right)}}{\frac{L}{C}} \\
&\times \quad \exp\left[-j\tan^{-1}\left(\frac{\omega L \Delta x \sqrt{\frac{L}{C} - \frac{\omega^2 L^2\Delta x^2}{4}}}{\frac{L}{C} - \frac{\omega^2 L^2 \Delta x^2}{2}}\right)\right]
\end{aligned}
$$

We increase the number of sections, $N \to \infty$, and decrease the length of each section, $\Delta x \to 0$, in such a way that the product $l = \Delta x\, N$ is kept constant. Hence, we have:

$$
\lim_{\substack{k\to\infty \\ \Delta x \to 0}} k\,\Delta x = x
$$

where x is now a continuous variable representing the physical length. $H(f, x)$ can be written as:

$$
\begin{aligned}
\lim_{\substack{k\to\infty \\ \Delta x \to 0}} H(f, k\,\Delta x) &= \lim_{\substack{k\to\infty \\ \Delta x \to 0}} \left(\frac{V(k\Delta x)}{V([k-1]\Delta x)}\right)^k \\
&= \lim_{\substack{k\to\infty \\ \Delta x \to 0}} \left(\frac{\sqrt{\left(\frac{L}{C}\right)^2}}{\frac{L}{C}}\right)^k \\
&\times \quad \lim_{\substack{k\to\infty \\ \Delta x \to 0}} \exp\left[-j\tan^{-1}\left(\frac{\omega L \Delta x \sqrt{\frac{L}{C} - \frac{\omega^2 L^2\Delta x^2}{4}}}{\frac{L}{C} - \frac{\omega^2 L^2 \Delta x^2}{2}}\right) k\right] \\
&= 1 \times \lim_{\substack{k\to\infty \\ \Delta x \to 0}} \exp\left[-j\tan^{-1}\left(\frac{\omega L \Delta x \sqrt{\frac{L}{C} - \frac{\omega^2 L^2\Delta x^2}{4}}}{\frac{L}{C} - \frac{\omega^2 L^2 \Delta x^2}{2}}\right) k\right]
\end{aligned}
$$

Expanding the arc-tangent in a series as follows:

$$
\tan^{-1}(\alpha) = \alpha - \frac{1}{2}\alpha^3 - 24\alpha^5 + \ldots
$$

with

$$
\alpha = \frac{\omega L \Delta x \sqrt{\frac{L}{C} - \frac{\omega^2 L^2\Delta x^2}{4}}}{\frac{L}{C} - \frac{\omega^2 L^2 \Delta x^2}{2}}
$$

then, the only term of the series, multiplied by k, that does not vanish when $k \to \infty$ and $\Delta x \to 0$ is the first one, that is,

$$\lim_{\substack{k\to\infty \\ \Delta x \to 0}} \exp\left[-j\tan^{-1}\left(\frac{\omega L \Delta x\sqrt{\frac{L}{C}-\frac{\omega^2 L^2 \Delta x^2}{4}}}{\frac{L}{C}-\frac{\omega^2 L^2 \Delta x^2}{2}}\right) k\right] \quad = \quad \exp\left(-j\,\omega\sqrt{LC}\,x\right)$$

Solution of problem 7.2

The current in a particular section of the line $k\,\Delta x$ can be written as

$$
\begin{aligned}
I(k\,\Delta x) &= \frac{V(k\,\Delta x)}{Z_L} \\
&= \left(\frac{V(k\Delta x)}{V([k-1]\Delta x)}\right)^k \frac{V(0)}{Z_L} \\
&= \left(\frac{V(k\Delta x)}{V([k-1]\Delta x)}\right)^k \frac{V(0)}{j\omega L\Delta x + \sqrt{\frac{L}{C} - \omega^2 \frac{L^2 \Delta x^2}{4}}}
\end{aligned}
$$

Taking the limits $N \to \infty$ and $\Delta x \to 0$ and using the results of the previous problem we have:

$$
I(x) = \frac{e^{-j\,\omega\,\sqrt{L\,C}\,x}}{\sqrt{\frac{L}{C}}}
$$

Solution of problem 7.3

$$
\begin{aligned}
Z_{in}(d) &= \frac{V(d)}{I(d)} \\
&= Z_o \frac{1+\Gamma(d)}{1-\Gamma(d)} \\
&= Z_o \frac{1+\Gamma_o\, e^{-2\,j\,\beta\,d}}{1-\Gamma_o\, e^{-2\,j\,\beta\,d}} \\
&= Z_o \frac{1+\frac{Z_L-Z_o}{Z_L+Z_o}\, e^{-2\,j\,\beta\,d}}{1-\frac{Z_L-Z_o}{Z_L+Z_o}\, e^{-2\,j\,\beta\,d}} \\
&= Z_o \frac{(Z_L+Z_o)\, e^{j\,\beta\,d}+(Z_L-Z_o)\, e^{-j\,\beta\,d}}{(Z_L+Z_o)\, e^{j\,\beta\,d}-(Z_L-Z_o)\, e^{-j\,\beta\,d}} \times \frac{e^{-j\,\beta\,d}}{e^{-j\,\beta\,d}} \\
&= Z_o \frac{Z_L\left(e^{j\,\beta\,d}+e^{-j\,\beta\,d}\right)+Z_o\left(e^{j\,\beta\,d}-e^{-j\,\beta\,d}\right)}{Z_o\left(e^{j\,\beta\,d}+e^{-j\,\beta\,d}\right)+Z_L\left(e^{j\,\beta\,d}-e^{-j\,\beta\,d}\right)} \\
&= Z_o \frac{Z_L\,\cos(\beta\,d)+j\,Z_o\,\sin(\beta\,d)}{Z_o\,\cos(\beta\,d)+j\,Z_L\,\sin(\beta\,d)} \\
&= Z_o \frac{Z_L+j\,Z_o\,\tan(\beta\,d)}{Z_o+j\,Z_L\,\tan(\beta\,d)}
\end{aligned}
$$

Solution of problem 7.4

The characteristic impedance is

$$\begin{aligned} Z_o &= \sqrt{\frac{L}{C}} \\ &= 74.2 \ \Omega \end{aligned}$$

For $\omega = 2\,\pi\,300$ rad/s we have:

$$\begin{aligned} \beta &= \omega\sqrt{L\,C} \\ &= 1.4\times 10^{-5} \quad \text{rad/m} \end{aligned}$$

and

$$\begin{aligned} Z_{in}(d=l) &= Z_o \frac{Z_L + j\,Z_o\,\tan(\beta\,l)}{Z_o + j\,Z_L\,\tan(\beta\,l)} \\ &= 25 + j\,0.01 \ \Omega \end{aligned}$$

In other words, the line impedance barely affects the input impedance making $Z_{in}(d=l) \simeq Z_L$.

For $\omega = 2\,\pi\,5\times 10^8$ rad/s we have:

$$\begin{aligned} \beta &= \omega\sqrt{L\,C} \\ &= 23.3 \quad \text{rad/m} \end{aligned}$$

and

$$\begin{aligned} Z_{in}(d=l) &= Z_o \frac{Z_L + j\,Z_o\,\tan(\beta\,l)}{Z_o + j\,Z_L\,\tan(\beta\,l)} \\ &= 137.5 + j\,96.3 \ \Omega \end{aligned}$$

At this high frequency the input impedance is greatly changed with respect to the load impedance.

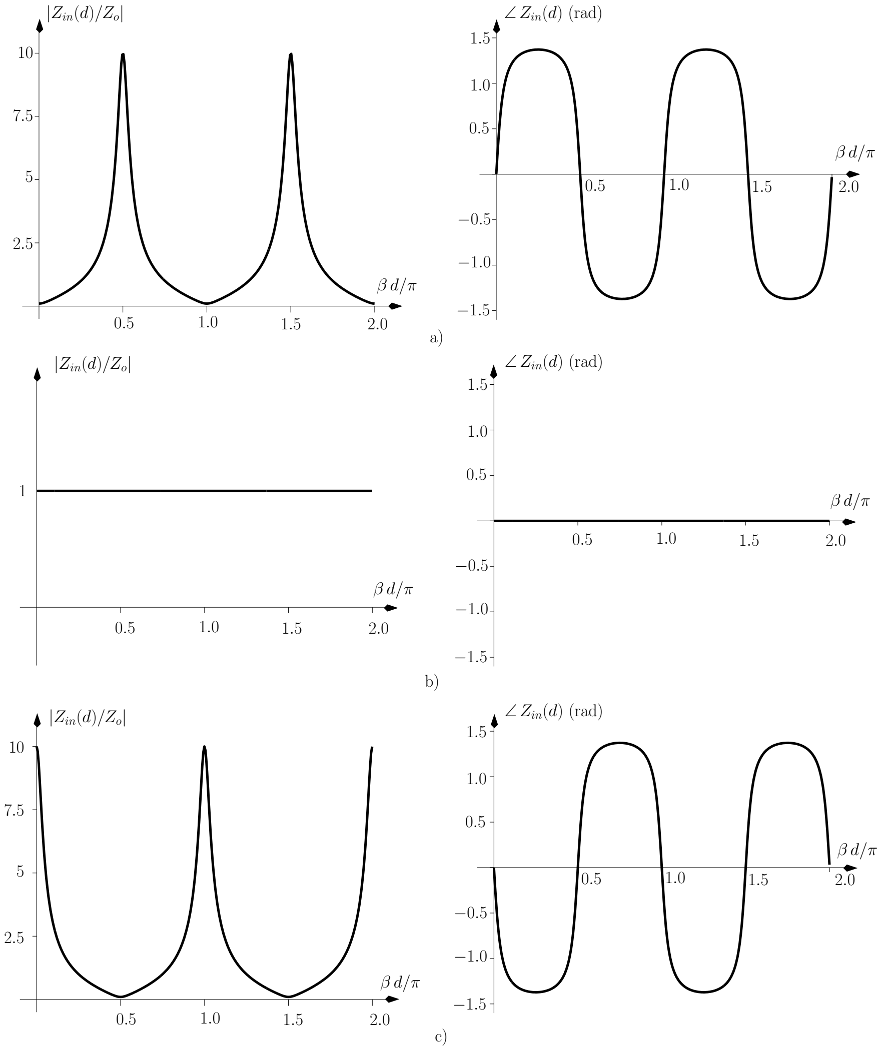


Figure 7.1: *Magnitude and angle of $Z_{in}(d)/Z_o$; a) $Z_L = Z_o/10$. b) $Z_L = Z_o$. c) $Z_L = 10\,Z_o$.*

Solution of problem 7.5

The input impedance normalised to the characteristic impedance can be written as

$$\begin{aligned}\frac{Z_{in}(d)}{Z_o} &= \frac{Z_L + j\,Z_o\,\tan(\beta\,l)}{Z_o + j\,Z_L\,\tan(\beta\,l)}\\ &= \frac{\frac{Z_L}{Z_o} + j\,\tan(\beta\,l)}{1 + j\,\frac{Z_L}{Z_o}\,\tan(\beta\,l)}\end{aligned}$$

Figure 7.1 a) shows the magnitude and angle of $Z_{in}(d)/Z_o$ when $Z_L = Z_o/10$. It can be seen that for $0 < \beta\,d < \pi/2$ and for $\pi < \beta\,d < 3\pi/2$ the input impedance features an inductive component while for $\pi/2 < \beta\,d < \pi$ and for $3\pi/2 < \beta\,d < 2\pi$ the input impedance features a capacitive component.

Figure 7.1 b) shows the magnitude and angle of $Z_{in}(d)/Z_o$ when $Z_L = Z_o$. Since the line is matched the input impedance is Z_o regardless of the value for $\beta\,d$.

Figure 7.1 c) shows the magnitude and angle of $Z_{in}(d)/Z_o$ when $Z_L = 10\,Z_o$. It can be seen that for $0 < \beta\,d < \pi/2$ and for $\pi < \beta\,d < 3\pi/2$ the input impedance is capacitive while for $\pi/2 < \beta\,d < \pi$ and for $3\pi/2 < \beta\,d < 2\pi$ the input impedance is inductive.

Solution of problem 7.6

The characteristic impedance of the quarter-wave transform must be:

$$\begin{aligned} Z_o &= \sqrt{30 \times 50} \\ &= 38.7\ \Omega \end{aligned}$$

Solution of problem 7.7

$$\begin{aligned}\Gamma_{tot} &= \frac{\Gamma_{AB}(1+\Gamma_{BL}\Gamma_{BA}) - \Gamma_{BL}T_{AB}T_{BA}}{1+\Gamma_{BL}\Gamma_{BA}} \\ &= \frac{\frac{Z_{oB}-Z_{oA}}{Z_{oB}+Z_{oA}}\left(1+\frac{Z_L-Z_{oB}}{Z_L+Z_{oB}}\,\frac{Z_{oA}-Z_{oB}}{Z_{oB}+Z_{oA}}\right)}{1+\frac{Z_L-Z_{oB}}{Z_L+Z_{oB}}\,\frac{Z_{oA}-Z_{oB}}{Z_{oB}+Z_{oA}}} - \frac{\frac{Z_L-Z_{oB}}{Z_L+Z_{oB}}\,\frac{2\,Z_{oB}}{Z_{oA}+Z_{oB}}\,\frac{2\,Z_{oA}}{Z_{oA}+Z_{oB}}}{1+\frac{Z_L-Z_{oB}}{Z_L+Z_{oB}}\,\frac{Z_{oA}-Z_{oB}}{Z_{oB}+Z_{oA}}}\end{aligned}$$

Simplifying this eqn we obtain

$$\Gamma_{tot} = \frac{Z_{oB}^2 - Z_{oA}\,Z_L}{Z_{oB}^2 + Z_{oA}\,Z_L}$$

Solution of problem 7.8

The current in a particular section of the line $k\,\Delta x$ can be written as

$$\begin{aligned} I(k\,\Delta x) &= \frac{V(k\,\Delta x)}{Z_L} \\ &= \left(\frac{V(k\Delta x)}{V([k-1]\Delta x)}\right)^k \frac{V(0)}{Z_L} \end{aligned}$$

with Z_L given by (see example 7.3.6):

$$Z_L = \frac{(R+j\,\omega\,L)\,\Delta x}{2} + \frac{\sqrt{(R+j\,\omega\,L)^2\,\Delta x^2 + 4\frac{R+j\,\omega\,L}{G+j\,\omega\,C}}}{2}$$

Taking the limits $N \to \infty$ and $\Delta x \to 0$ and using the results derived in example 7.3.6 we obtain:

$$I(x) = \frac{e^{-j\,\omega\,\sqrt{L\,C}\,x}}{\sqrt{\frac{R+j\,\omega\,L}{G+j\,\omega\,C}}}$$

and since $V(x) = I(x)\,Z_o$ we can write:

$$Z_o = \sqrt{\frac{R+j\,\omega\,L}{G+j\,\omega\,C}}$$

Solution of problem 7.9

1. The complex propagation constant for a lossy transmission line can be expressed as:

$$\begin{aligned}\gamma &= \sqrt{RG\left(1+\frac{j\omega L}{R}\right)\left(1+\frac{j\omega C}{G}\right)} \\ \gamma &= \sqrt{RG}\sqrt{1+j\left(\frac{\omega L}{R}+\frac{\omega C}{G}\right)-\frac{\omega^2 LC}{RG}}\end{aligned} \tag{7.1}$$

   If $R >> \omega L$ and $G >> \omega C$ we have

$$1 >> \left(\frac{\omega L}{R}+\frac{\omega C}{G}\right) >> \frac{\omega^2 LC}{RG}$$

   and we can write:

$$\gamma \simeq \sqrt{RG}\sqrt{1+j\left(\frac{\omega L}{R}+\frac{\omega C}{G}\right)}$$

   Taking into account that $\sqrt{1+x} \simeq 1+x/2$ if $x << 1$ we can write:

$$\begin{aligned}\gamma &\simeq \sqrt{RG}\left[1+j\left(\frac{\omega L}{R}+\frac{\omega C}{G}\right)\right] \\ &= \underbrace{\sqrt{RG}}_{\alpha_{LF}}+j\underbrace{\omega\frac{1}{2}\left(C\sqrt{\frac{R}{G}}+L\sqrt{\frac{G}{R}}\right)}_{\beta_{LF}}\end{aligned}$$

   If $R >> \omega L$ and $G >> \omega C$ we can also write

$$\begin{aligned}Z_o &= \sqrt{\frac{R+j\omega L}{G+j\omega C}} \\ &\simeq \sqrt{\frac{R}{G}}\end{aligned}$$

2. The complex propagation constant for a lossy transmission line can also be expressed as:

$$\begin{aligned}\gamma &= \sqrt{(j\omega L)(j\omega C)\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)} \\ &= j\omega\sqrt{LC}\sqrt{1-j\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)-\frac{RG}{\omega^2 LC}}\end{aligned}$$

   If $R << \omega L$ and $G << \omega C$ we have

$$1 >> \left(\frac{R}{\omega L}+\frac{G}{\omega C}\right) >> \frac{RG}{\omega^2 LC}$$

   and we can write:

$$\begin{aligned}\gamma &\simeq j\omega\sqrt{LC}\sqrt{1-j\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)} \\ &\simeq j\omega\sqrt{LC}\left[1-j\frac{1}{2}\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)\right] \\ &= \underbrace{\frac{1}{2}\left(R\sqrt{\frac{C}{L}}+G\sqrt{\frac{L}{C}}\right)}_{\alpha_{HF}}+j\underbrace{\omega\sqrt{LC}}_{\beta_{HF}}\end{aligned}$$

For $R << \omega\, L$ and $G << \omega\, C$ we have

$$\begin{aligned} Z_o &= \sqrt{\frac{R + j\,\omega\, L}{G + j\,\omega\, C}} \\ &\simeq \sqrt{\frac{j\,\omega\, L}{j\,\omega\, C}} \\ &= \sqrt{\frac{L}{C}} \end{aligned}$$

Solution of problem 7.10

$$
\begin{aligned}
Z_{in}(d) &= \frac{V(d)}{I(d)} \\
&= Z_o \frac{1+\Gamma(d)}{1-\Gamma(d)} \\
&= Z_o \frac{1+\Gamma_o\, e^{-2\,\gamma\, d}}{1-\Gamma_o\, e^{-2\,\gamma\, d}} \\
&= Z_o \frac{1+\frac{Z_L-Z_o}{Z_L+Z_o}\, e^{-2\,\gamma\, d}}{1-\frac{Z_L-Z_o}{Z_L+Z_o}\, e^{-2\,\gamma\, d}} \\
&= Z_o \frac{(Z_L+Z_o)\, e^{\gamma\, d}+(Z_L-Z_o)\, e^{-\gamma\, d}}{(Z_L+Z_o)\, e^{\gamma\, d}-(Z_L-Z_o)\, e^{-\gamma\, d}} \times \frac{e^{-\gamma\, d}}{e^{-\gamma\, d}} \\
&= Z_o \frac{Z_L\left(e^{\gamma\, d}+e^{-\gamma\, d}\right)+Z_o\left(e^{\gamma\, d}-e^{-\gamma\, d}\right)}{Z_o\left(e^{\gamma\, d}+e^{-\gamma\, d}\right)+Z_L\left(e^{\gamma\, d}-e^{-\gamma\, d}\right)} \\
&= Z_o \frac{Z_L\, \cosh(\gamma\, d)+Z_o\, \sinh(\gamma\, d)}{Z_o\, \cosh(\gamma\, d)+Z_L\, \sinh(\gamma\, d)} \\
&= Z_o \frac{Z_L+Z_o\, \tanh(\gamma\, d)}{Z_o+Z_L\, \tanh(\gamma\, d)}
\end{aligned}
$$

Solution of problem 7.11

assuming that $\frac{W}{d} < 2$ we have that

$$\begin{aligned} \frac{W}{d} &\simeq \frac{8\, e^{A}}{e^{2A} - 2} \\ &= 1.164 \end{aligned}$$

where

$$\begin{aligned} A &= \frac{Z_o}{60}\sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1}\left(0.23 + \frac{0.11}{\epsilon_r}\right) \\ &= 1.968 \end{aligned}$$

Hence, we have:

$$\begin{aligned} W &= d \times 1.164 \\ &= 1.51 \ \ \text{mm} \end{aligned}$$

Solution of problem 7.12

Figure 7.2 a) shows the equivalent circuit for the calculation of S_{11} and S_{21}. S_{11} is calculated as

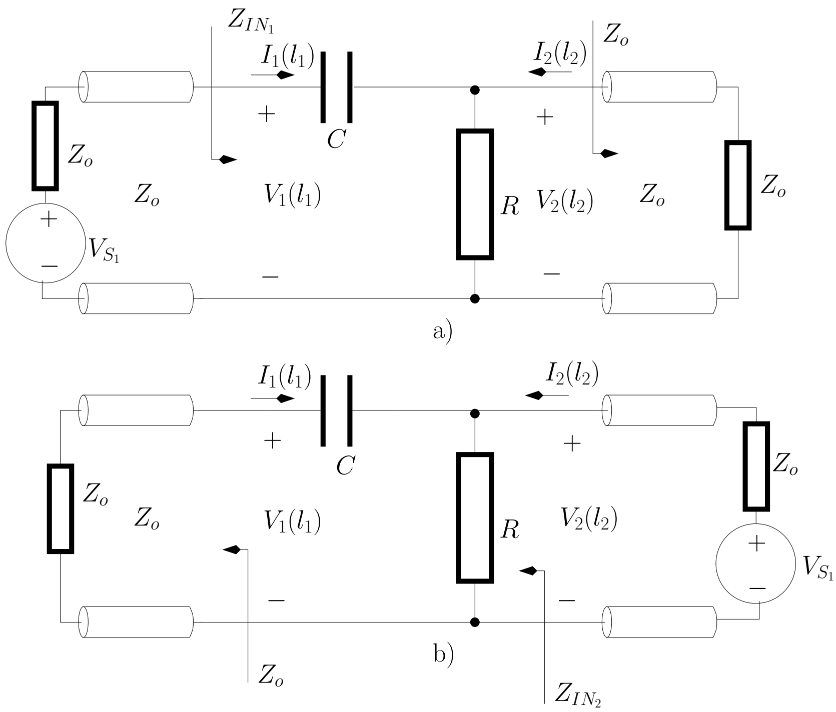


Figure 7.2: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

$$S_{11} = \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_1} &= \frac{1}{j\,\omega\,C} + (R\,||\,Z_o) \\ &= \frac{R + Z_o + j\,\omega\,C\,R\,Z_o}{j\,\omega\,C\,(R + Z_o)} \end{aligned}$$

and S_{11} can be written as

$$S_{11} = \frac{R + Z_o\,(1 - Z_o\,j\,\omega\,C)}{2\,j\,\omega\,C\,R\,Z_o + R + Z_o\,(1 + j\,\omega\,C\,Z_o)}$$

S_{21} is calculated as

$$\begin{aligned} S_{21} &= \left.\frac{b_2(l_2)}{a_1(l_1)}\right|_{a_2(l_2)=0} \\ &= \frac{V_2(l_2) - Z_o\,I_2(l_2)}{V_1(l_1) + Z_o\,I_1(l_1)} \end{aligned}$$

It is known that:

$$\begin{aligned} V_1(l_1) &= Z_{IN_1}\,I_1(l_1) \\ &= \frac{R + Z_o + j\,\omega\,C\,R\,Z_o}{j\,\omega\,C\,(R + Z_o)}\,I_1(l_1) \end{aligned}$$

Also, $V_2(l_2)$ can be related to $V_1(l_1)$ by the voltage divider expression:

$$\begin{aligned} V_2(l_2) &= \frac{R\,||\,Z_o}{\frac{1}{j\,\omega\,C} + (R\,||\,Z_o)}\,V_1(l_1) \\ &= \frac{j\,\omega\,C\,R\,Z_o}{j\,\omega\,C\,R\,Z_o + R + Z_o} \end{aligned}$$

Now, $a_2(l_2) = 0$ implying that $V_2(l_2) = -Z_o\,I_2(l_2)$ and we have

$$\begin{aligned} S_{21} &= \frac{2\,V_2(l_2)}{V_1(l_1)\,\left(1 + \frac{Z_o}{Z_{IN_1}}\right)} \\ &= \frac{2\,j\,\omega\,C\,R\,Z_o}{2\,j\,\omega\,C\,R\,Z_o + R + Z_o\,(1 + j\,\omega\,C\,Z_o)} \end{aligned}$$

Figure 7.2 b) shows the equivalent circuit for the calculation of S_{22} and S_{12}. S_{22} is calculated as

$$S_{22} = \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_2} &= \left(\frac{1}{j\,\omega\,C} + Z_o\right)\,||\,R \\ &= \frac{R\,(1 + j\,\omega\,C\,Z_o)}{1 + j\,\omega\,C\,(Z_o + R)} \end{aligned}$$

and S_{22} is

$$S_{22} = \frac{R - Z_o\,(1 + j\,\omega\,C\,Z_o)}{2\,j\,\omega\,C\,Z_o\,R + R + Z_o\,(1 + j\,\omega\,C\,Z_o)}$$

S_{12} is given by

$$\begin{aligned} S_{12} &= \left.\frac{b_1(l_1)}{a_2(l2)}\right|_{a_1(l1)=0} \\ &= \frac{V_1(l_1) - Z_o I_1(l_1)}{V_2(l_2) + Z_o I_2(l_2)} \end{aligned} \qquad (7.2)$$

Since $a_1(l_1) = 0 \;\Rightarrow\; V_1(l_1) = -Z_o\,I_1(l_1)$, the last eqn can be written as

$$S_{12} = \frac{2\,V_1(l_1)}{V_2(l_2) + Z_o I_2(l_2)} \qquad (7.3)$$

$V_1(l_1)$ can be related to $V_2(l_2)$ using the impedance voltage divider formula:

$$V_1(l_1) = V_2(l_2)\,\frac{j\,\omega\,C\,Z_o}{1 + j\,\omega\,C\,Z_o}$$

Using the result of the last eqn and since $V_2(l_2) = Z_{IN_2}\,I_2(l_2)$, we can calculate S_{12} as follows:

$$\begin{aligned} S_{12} &= \frac{2\,V_1(l_1)}{V_2(l_2)\left(1 + \frac{Z_o}{Z_{IN_2}}\right)} \\ &= \frac{2\,j\,\omega\,C\,Z_o\,R}{2\,j\,\omega\,C\,Z_o\,R + R + Z_o\,(1 + j\,\omega\,C\,Z_o)} \end{aligned}$$

Solution of problem 7.13

- *circuit a):* Figure 7.3 a) shows the circuit for the calculation of S_{11} and S_{21}. S_{11} is calculated

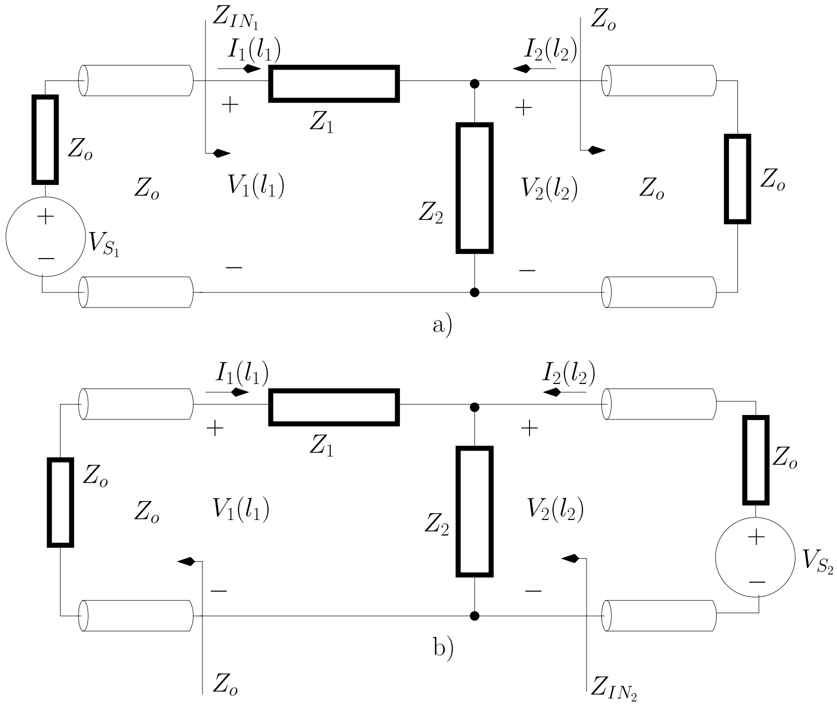


Figure 7.3: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

as

$$S_{11} = \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_1} &= Z_1 + (Z_2 \,||\, Z_o) \\ &= \frac{Z_1(Z_2 + Z_o) + Z_2\, Z_o}{Z_2 + Z_o} \end{aligned}$$

and S_{11} is

$$S_{11} = \frac{Z_1\, Z_2 + Z_o\,(Z_1 - Z_o)}{Z_o\, Z_2 + (Z_2 + Z_o)\,(Z_1 + Z_o)}$$

S_{21} is calculated as

$$\begin{aligned} S_{21} &= \left.\frac{b_2(l_2)}{a_1(l_1)}\right|_{a_2(l_2)=0} \\ &= \frac{V_2(l_2) - Z_o\, I_2(l_2)}{V_1(l_1) + Z_o\, I_1(l_1)} \end{aligned}$$

$a_2(l_2) = 0$ implying that $V_2(l_2) = -Z_o\, I_2(l_2)$. Also, it is known that:

$$V_1(l_1) = Z_{IN_1}\, I_1(l_1)$$

Hence, we have

$$S_{21} = \frac{2\, V_2(l_2)}{V_1(l_1)\,\left(1 + \frac{Z_o}{Z_{IN_1}}\right)}$$

$V_2(l_2)$ can be related to $V_1(l_1)$ by the voltage divider expression:

$$V_2(l_2) = \frac{Z_o\,Z_2}{Z_o\,Z_2 + Z_1\,(Z_2 + Z_o)}\,V_1(l_1)$$

therefore, we have

$$S_{21} = \frac{2\,Z_o\,Z_2}{Z_o\,Z_2 + (Z_2 + Z_o)\,(Z_1 + Z_o)}$$

Figure 7.3 b) shows the circuit for the calculation of S_{22} and S_{12}. S_{22} is calculated as

$$\begin{aligned} S_{22} &= \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o} \\ &= \frac{Z_1\,Z_2 - Z_o(Z_1 + Z_o)}{Z_o\,Z_2 + (Z_2 + Z_o)\,(Z_1 + Z_o)} \end{aligned}$$

with:

$$Z_{IN_2} = \frac{(Z_o + Z_1)\,Z_2}{Z_o + Z_1 + Z_2}$$

S_{12} is given by

$$\begin{aligned} S_{12} &= \left.\frac{b_1(l_1)}{a_2(l2)}\right|_{a_1(l1)=0} \\ &= \frac{V_1(l_1) - Z_o I_1(l_1)}{V_2(l_2) + Z_o I_2(l_2)} \qquad (7.4) \end{aligned}$$

Since $a_1(l_1) = 0 \;\Rightarrow\; V_1(l_1) = -Z_o\,I_1(l_1)$, and since $V_2(l_2) = Z_{IN2}\,I_2(l_2)$ the last eqn can be written as

$$S_{12} = \frac{2\,V_1(l_1)}{V_2(l_2)\left(1 + \frac{Z_o}{Z_{IN_2}}\right)} \qquad (7.5)$$

$V_1(l_1)$ can be related to $V_2(l_2)$ using the impedance voltage divider formula:

$$V_1(l_1) = V_2(l_2)\,\frac{Z_o}{Z_o + Z_1}$$

Using the result of the last eqn calculate S_{12} as follows:

$$S_{12} = \frac{2\,Z_o\,Z_2}{Z_o\,Z_2 + (Z_2 + Z_o)\,(Z_1 + Z_o)}$$

- *circuit b):* Figure 7.4 a) shows the circuit for the calculation of S_{11} and S_{21}. S_{11} is calculated as

$$\begin{aligned} S_{11} &= \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o} \\ &= \frac{Z_1\,Z_2 - Z_o\,(Z_2 + Z_o)}{Z_1\,Z_o + (Z_o + Z_1)\,(Z_o + Z_2)} \end{aligned}$$

with:

$$Z_{IN_1} = \frac{Z_1\,(Z_2 + Z_o)}{Z_1 + Z_2 + Z_o}$$

S_{21} is calculated as

$$\begin{aligned} S_{21} &= \frac{2\,V_2(l_2)}{V_1(l_1)\left(1 + \frac{Z_o}{Z_{IN_1}}\right)} \\ &= \frac{2\,Z_1\,Z_o}{Z_1\,Z_o + (Z_o + Z_1)\,(Z_o + Z_2)} \end{aligned}$$

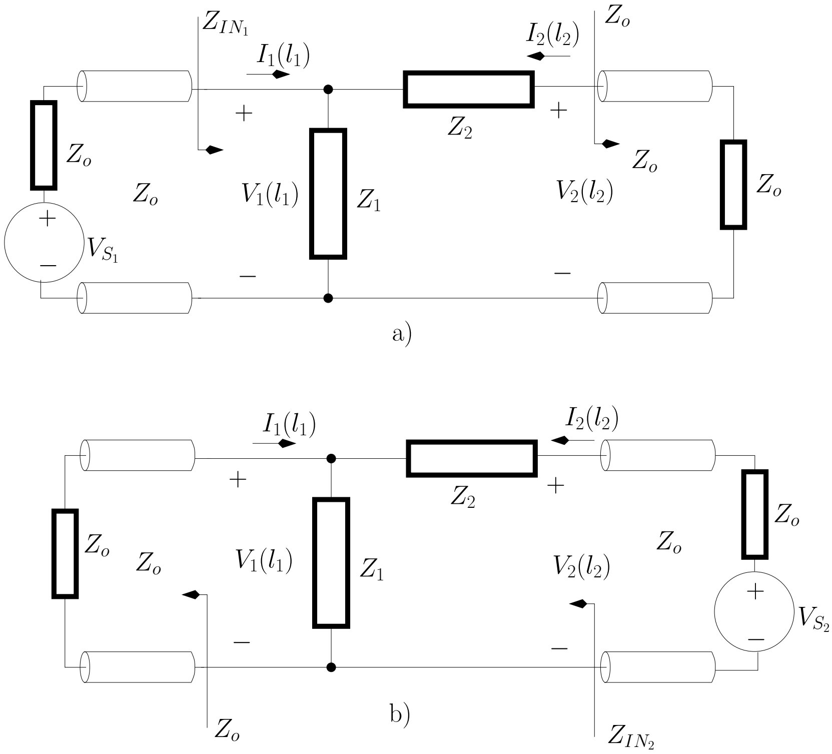


Figure 7.4: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

Figure 7.4 b) shows the circuit for the calculation of S_{22} and S_{12}.

$$\begin{aligned} S_{22} &= \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o} \\ &= \frac{Z_1\,Z_2 + Z_o\,(Z_2 - Z_o)}{Z_o\,Z_1 + (Z_2 + Z_o)\,(Z_1 + Z_o)} \end{aligned}$$

with:

$$Z_{IN_2} = \frac{Z_2\,(Z_1 + Z_o) + Z_1\,Z_o}{Z_1 + Z_o}$$

S_{12} is given by

$$\begin{aligned} S_{12} &= \frac{2\,V_1(l_1)}{V_2(l_2)\left(1 + \frac{Z_o}{Z_{IN_2}}\right)} \\ &= \frac{2\,Z_1\,Z_o}{Z_1\,Z_o + (Z_o + Z_1)\,(Z_o + Z_2)} \end{aligned}$$

with

$$\frac{V_1(l_1)}{V_2(l_2)} = \frac{Z_1\,Z_o}{Z_2\,(Z_1 + Z_o) + Z_1\,Z_o}$$

- *circuit c):* Figure 7.5 a) shows the circuit for the calculation of S_{11} and S_{21}. For this circuit we can write:

$$V_2(l_2) = A_{fi}\,I_1(l_1)\,(R_o\,||\,Z_o)$$

and

$$\begin{aligned} V_1(l_1) &= I_1(l_1)\,R_i + A_{rv}\,V_2(l_2) \\ &= I_1(l_1)[R_i + A_{rv}\,A_{fi}\,(R_o\,||\,Z_o)] \end{aligned}$$

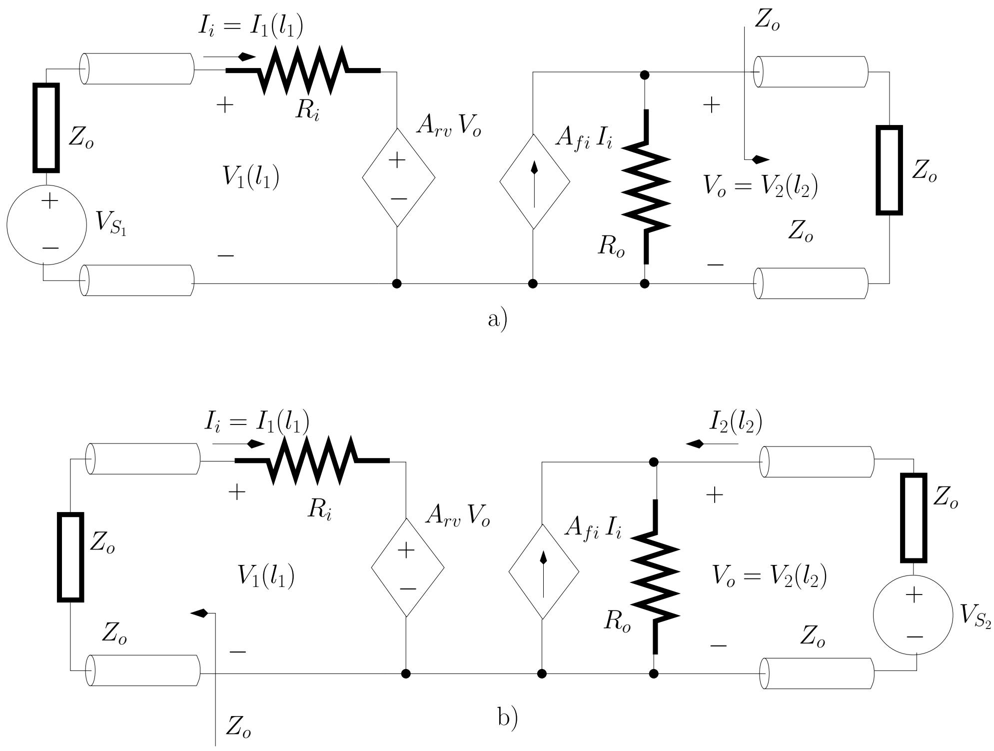


Figure 7.5: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

Now, S_{11} can be calculated as:

$$
\begin{aligned}
S_{11} &= \left.\frac{b_1(l_1)}{a_1(l_1)}\right|_{a_2(l_2)=0} \\
&= \frac{V_1(l_1) - Z_o\,I_1(l_1)}{V_1(l_1) + Z_o\,I_1(l_1)} \\
&= \frac{R_i + A_{rv}\,A_{fi}\,\,(R_o\,||\,Z_o) - Z_o}{R_i + A_{rv}\,A_{fi}\,\,(R_o\,||\,Z_o) + Z_o} \\
&= \frac{(R_i - Z_o)\,(R_o + Z_o) + A_{rv}\,A_{fi}\,\,R_o\,Z_o}{(R_i + Z_o)\,(R_o + Z_o) + A_{rv}\,A_{fi}\,\,R_o\,Z_o}
\end{aligned}
$$

S_{21} can be calculated as:

$$
\begin{aligned}
S_{21} &= \left.\frac{b_2(l_2)}{a_1(l_1)}\right|_{a_2(l_2)=0} \\
&= \frac{2\,V_2(l_2)}{V_1(l_1) + Z_o\,I_1(l_1)} \\
&= \frac{2\,A_{fi}\,\,R_o\,Z_o}{(R_i + Z_o)\,(R_o + Z_o) + A_{rv}\,A_{fi}\,\,R_o\,Z_o}
\end{aligned}
$$

Figure 7.5 b) shows the circuit for the calculation of S_{22} and S_{12}.
We can write the following eqns:

$$V_1(l_1) = \frac{Z_o}{Z_o + R_i}\,A_{rv}\,V_2(l_2) \tag{7.6}$$

$$V_1(l_1) = I_1(l_1)\,R_i + A_{rv}\,V_2(l_2) \tag{7.7}$$

$$I_2(l_2) = \frac{V_2(l_2)}{R_o} - A_{fi}\,I_1(l_1) \tag{7.8}$$

From eqns 7.6 and 7.7 we have

$$I_1(l_1) \quad = \quad -V_2(l_2)\frac{A_{rv}}{Z_o + R_i} \tag{7.9}$$

and now we can write eqn 7.8 as follows.

$$I_2(l_2) \quad = \quad \frac{V_2(l_2)}{R_o} + A_{fi}\, V_2(l_2)\frac{A_{rv}}{Z_o + R_i}$$

S_{22} can be calculated as:

$$\begin{aligned} S_{22} &= \left.\frac{b_2(l_2)}{a_2(l_2)}\right|_{a_1(l_1)=0} \\ &= \frac{V_2(l_2) - Z_o\, I_2(l_2)}{V_2(l_2) + Z_o\, I_2(l_2)} \\ &= \frac{(R_i + Z_o)\,(R_o - Z_o) - A_{rv}\, A_{fi}\; R_o\, Z_o}{(R_i + Z_o)\,(R_o + Z_o) + A_{rv}\, A_{fi}\; R_o\, Z_o} \end{aligned}$$

S_{12} can be calculated as:

$$\begin{aligned} S_{12} &= \left.\frac{b_1(l_1)}{a_2(l_2)}\right|_{a_1(l_1)=0} \\ &= \frac{2\, V_1(l_1)}{V_2(l_2) + Z_o\, I_2(l_2)} \\ &= \frac{2\, A_{rv}\; R_o\, Z_o}{(R_i + Z_o)\,(R_o + Z_o) + A_{rv}\, A_{fi}\; R_o\, Z_o} \end{aligned}$$

- *circuit d):* Figure 7.6 a) shows the circuit for the calculation of S_{11} and S_{21}. S_{11} can be

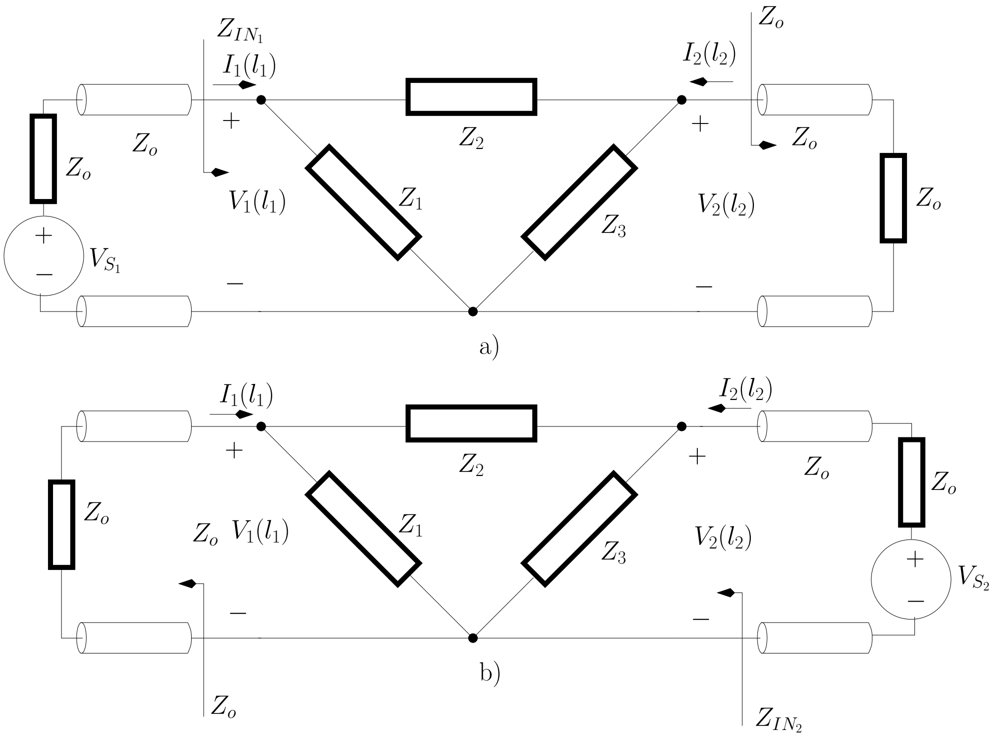


Figure 7.6: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

calculated from the following eqn:

$$S_{11} \quad = \quad \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_1} &= [\,(Z_3 \,||\, Z_o) + Z_2\,] \,||\, Z_1 \\ &= Z_1 \frac{Z_o\,(Z_3 + Z_2) + Z_3\,Z_2}{Z_3\,(Z_1 + Z_2) + Z_o\,(Z_1 + Z_2 + Z_3)} \end{aligned}$$

S_{21} can be calculated from the following eqn

$$S_{21} = \frac{2\,V_2(l_2)}{V_1(l_1)\,\left(1 + \frac{Z_o}{Z_{IN_1}}\right)}$$

with

$$\frac{V_2(l_2)}{V_1(l_1)} = \frac{Z_3\,Z_o}{Z_3\,Z_o + Z_2\,(Z_3 + Z_o)}$$

Figure 7.6 b) shows the circuit for the calculation of S_{22} and S_{12}. S_{22} can be calculated from the following eqn:

$$S_{22} = \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_2} &= [\,(Z_1 \,||\, Z_o) + Z_2\,] \,||\, Z_3 \\ &= Z_3 \frac{Z_o\,(Z_1 + Z_2) + Z_1\,Z_2}{Z_1\,(Z_3 + Z_2) + Z_o\,(Z_3 + Z_2 + Z_1)} \end{aligned}$$

and S_{12} can be calculated from the following eqn

$$S_{12} = \frac{2\,V_1(l_1)}{V_2(l_2)\,\left(1 + \frac{Z_o}{Z_{IN_2}}\right)}$$

with

$$\frac{V_1(l_1)}{V_2(l_2)} = \frac{Z_1\,Z_o}{Z_1\,Z_o + Z_2\,(Z_1 + Z_o)}$$

- *circuit e):* Figure 7.7 a) shows the circuit for the calculation of S_{11} and S_{21}. S_{11} can be calculated from the following eqn:

$$S_{11} = \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_1} &= Z_1 + [Z_3 \,||\, (Z_2 + Z_o)] \\ &= Z_1 + \frac{Z_3\,(Z_2 + Z_o)}{Z_3 + Z_2 + Z_o} \end{aligned}$$

S_{21} can be calculated from the following eqn

$$S_{21} = \frac{2\,V_2(l_2)}{V_1(l_1)\,\left(1 + \frac{Z_o}{Z_{IN_1}}\right)}$$

with

$$\begin{aligned} \frac{V_2(l_2)}{V_1(l_1)} &= \frac{V_2(l_2)}{V'} \times \frac{V'}{V_1(l_1)} \\ &= \frac{Z_o}{Z_o + Z_2} \times \frac{Z_3 \,||\, (Z_2 + Z_o)}{Z_1 + [Z_3 \,||\, (Z_2 + Z_o)]} \\ &= \frac{Z_o}{Z_o + Z_2} \times \frac{Z_3\,(Z_2 + Z_o)}{Z_3\,(Z_2 + Z_o) + Z_1\,(Z_3 + Z_2 + Z_o)} \\ &= \frac{Z_3\,Z_o}{Z_3\,(Z_2 + Z_o) + Z_1\,(Z_3 + Z_2 + Z_o)} \end{aligned}$$

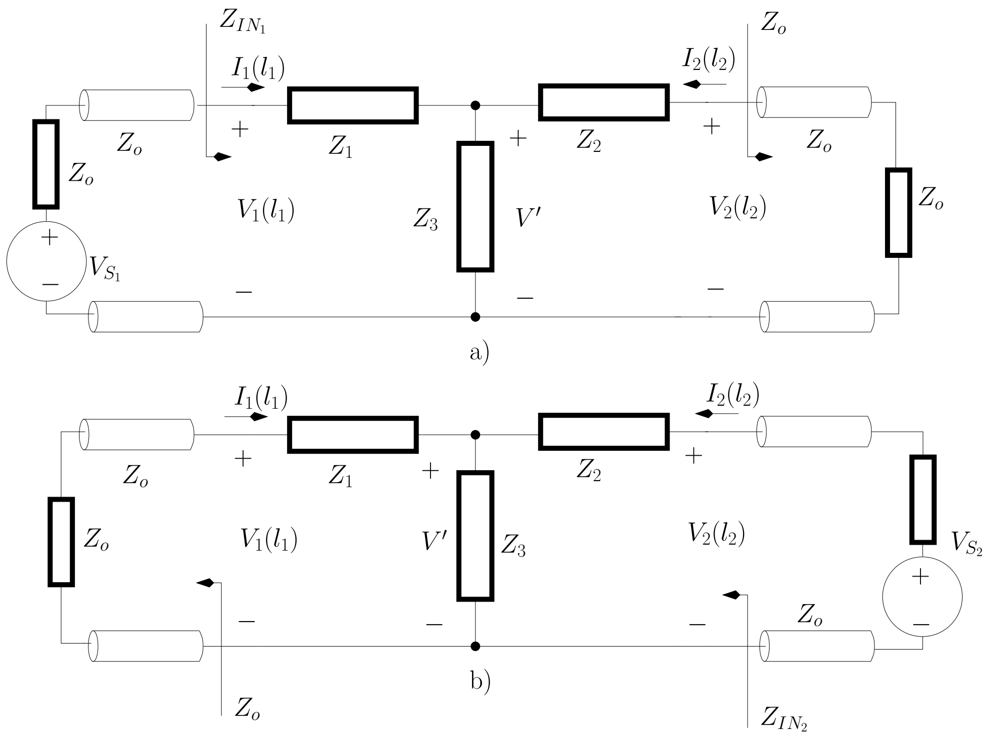


Figure 7.7: *a) Calculation of S_{11} and S_{21}. b) Calculation of S_{12} and S_{22}.*

Figure 7.7 b) shows the circuit for the calculation of S_{22} and S_{12}. S_{22} can be calculated from the following eqn:

$$S_{22} = \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o}$$

with:

$$\begin{aligned} Z_{IN_2} &= Z_2 + [Z_3 \,||\, (Z_1 + Z_o)] \\ &= Z_2 + \frac{Z_3\,(Z_1 + Z_o)}{Z_3 + Z_1 + Z_o} \end{aligned}$$

and S_{12} can be calculated from the following eqn

$$S_{12} = \frac{2\,V_1(l_1)}{V_2(l_2)\,\left(1 + \frac{Z_o}{Z_{IN_2}}\right)}$$

with

$$\begin{aligned} \frac{V_1(l_1)}{V_2(l_2)} &= \frac{V_1(l_1)}{V'} \times \frac{V'}{V_2(l_2)} \\ &= \frac{Z_o}{Z_o + Z_1} \times \frac{Z_3 \,||\, (Z_1 + Z_o)}{Z_2 + [Z_3 \,||\, (Z_1 + Z_o)]} \\ &= \frac{Z_3\,Z_o}{Z_3\,(Z_1 + Z_o) + Z_2\,(Z_3 + Z_1 + Z_o)} \end{aligned}$$

Solution of problem 7.14

Figure 7.8 a) shows the circuit for the calculation of S_{11} and S_{21}. S_{11} can be calculated from the

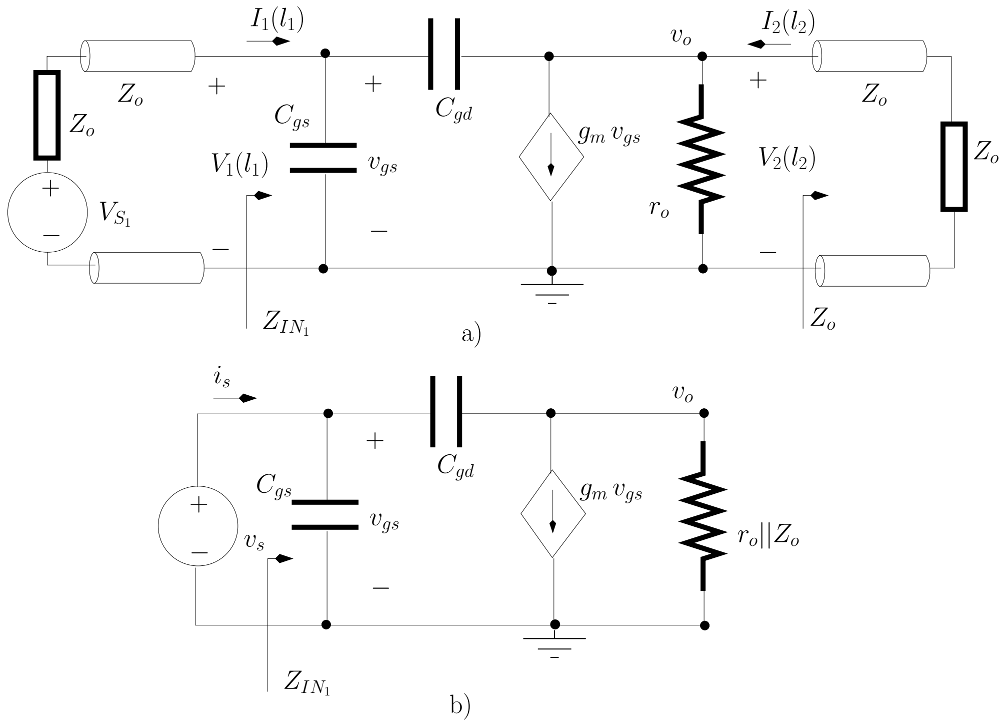


Figure 7.8: *a) Calculation of S_{11} and S_{21}. b) Calculation of $Z_{IN_1} = v_s/i_s$ and of $A_{v_1} = v_o/v_s$.*

following expression

$$S_{11} = \frac{Z_{IN_1} - Z_o}{Z_{IN_1} + Z_o} \tag{7.10}$$

where Z_{IN_1} can be calculated from the circuit of figure 7.8 b). For this circuit we can write the following set of eqns

$$\begin{cases} i_s = v_s\, j\,\omega\, C_{gs} + (v_s - v_o)\, j\,\omega\, C_{gd} \\ (v_s - v_o)\, j\,\omega\, C_{gd} = g_m\, v_s + v_o/(r_o||Z_o) \end{cases} \tag{7.11}$$

Solving to obtain v_s/i_s we get

$$Z_{IN_1} = \frac{r_o + Z_o + j\,\omega\, C_{gd}\, r_o\, Z_o}{j\,\omega\,[(C_{gd} + C_{gs})\,(r_o + Z_o) + C_{gd}\, g_m\, r_o\, Z_o] - \omega^2\, C_{gd}\, C_{gs}\, r_o\, Z_o} \tag{7.12}$$

From which S_{11} can be found.

S_{21} can be calculated from the following expression

$$S_{21} = \frac{2\, A_{v_1}}{1 + \frac{Z_o}{Z_{IN_1}}}$$

where A_{v_1} is given by

$$A_{v_1} = \frac{V_2(l_2)}{V_1(l_1)}$$

A_{v_1} can be obtained from the circuit of figure 7.8 b) as follows:

$$A_{v_1} = \frac{v_o}{v_s}$$

Solving again the set of eqns, given by 7.11, to obtain v_o/v_s we get:

$$A_{v_1} \quad = \quad \frac{r_o\, Z_o(j\,\omega\, C_{gd} - g_m)}{j\,\omega\, C_{gd}\, r_o\, Z_o + r_o + Z_o} \tag{7.13}$$

Figure 7.9 a) shows the circuit for the calculation of S_{12} and S_{22}. S_{22} can be calculated from the

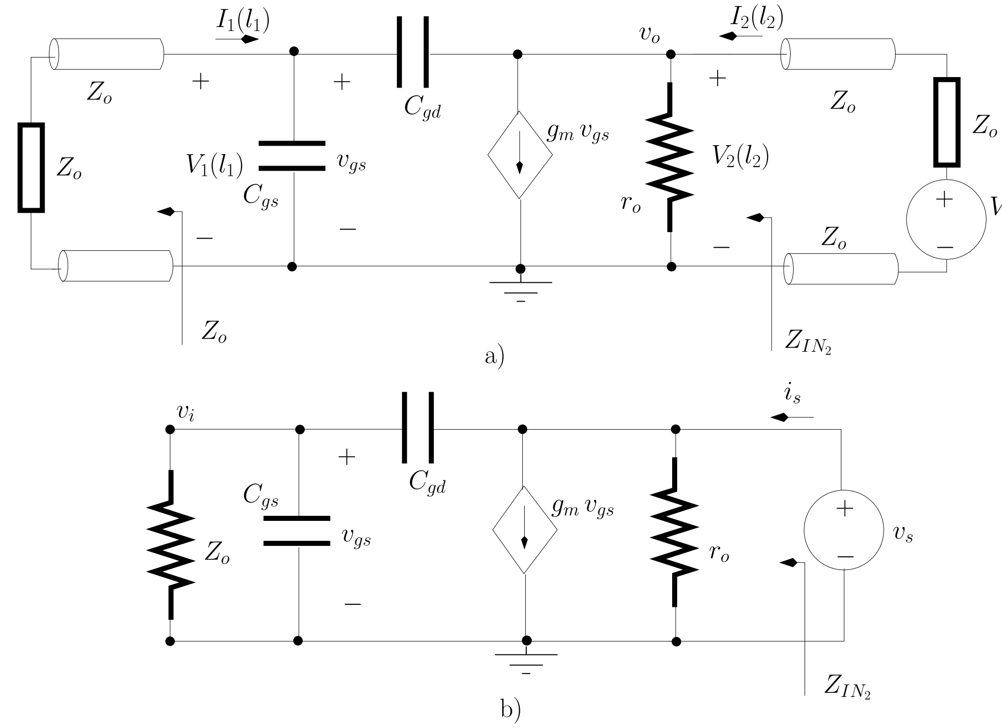


Figure 7.9: *a) Calculation of S_{22} and S_{12}. b) Calculation of $Z_{IN_2} = v_s/i_s$ and of $A_{v_2} = v_i/v_s$.*

following expression

$$S_{22} \quad = \quad \frac{Z_{IN_2} - Z_o}{Z_{IN_2} + Z_o} \tag{7.14}$$

where Z_{IN_2} can be calculated from the circuit of figure 7.9 b). For this circuit we can write the following set of eqns

$$\begin{cases} i_s = v_s/r_o + g_m\, v_i + (v_s - v_i)\, j\,\omega\, C_{gd} \\ \\ (v_s - v_i)\, j\,\omega\, C_{gd} = v_i\, j\,\omega\, C_{gs} + v_i/Z_o \end{cases} \tag{7.15}$$

Solving to obtain v_s/i_s we get

$$Z_{IN_2} \quad = \quad \frac{r_o[j\,\omega\,(C_{gd} + C_{gs})\, Z_o + 1]}{1 + j\,\omega\,(C_{gd} + C_{gs})\, Z_o + j\,\omega\, C_{gd}\, r_o\,(g_m\, Z_o + 1) - \omega^2\, C_{gd}\, C_{gs}\, r_o\, Z_o} \tag{7.16}$$

S_{12} can be calculated from the eqn below:

$$\begin{aligned} S_{12} \quad &= \quad \frac{2\, A_{v_2}}{1 + \frac{Z_o}{Z_{IN_2}}} \qquad (7.17) \\ &= \quad \frac{2\, Z_1\, Z_o}{Z_1\, Z_o + (Z_o + Z_1)\,(Z_o + Z_2)} \end{aligned}$$

with

$$A_{v_2} = \frac{V_1(l_1)}{V_2(l_2)}$$

A_{v_2} can be calculated by solving the set of eqns 7.15 (see also figure 7.9 b)) to obtain v_i/v_s

$$A_{v_2} = \frac{j\,\omega\,C_{gd}\,Z_o}{1 + j\,\omega\,(C_{gd} + C_{gs})\,Z_o} \tag{7.18}$$

Solution of problem 7.15

The transconductance can be calculated as follows:

$$\begin{aligned} g_m &= \sqrt{k_n \frac{W}{L} 2\, I_D} \\ &= 6.3 \ \text{mA/V} \end{aligned} \tag{7.19}$$

The drain-source resistance r_o can be determined as follows:

$$\begin{aligned} r_o &= \frac{V_A}{I_D} \\ &= 12 \ \text{k}\Omega \end{aligned} \tag{7.20}$$

We use eqns 7.10-7.18 to plot the S-parameters shown in figure 7.10.

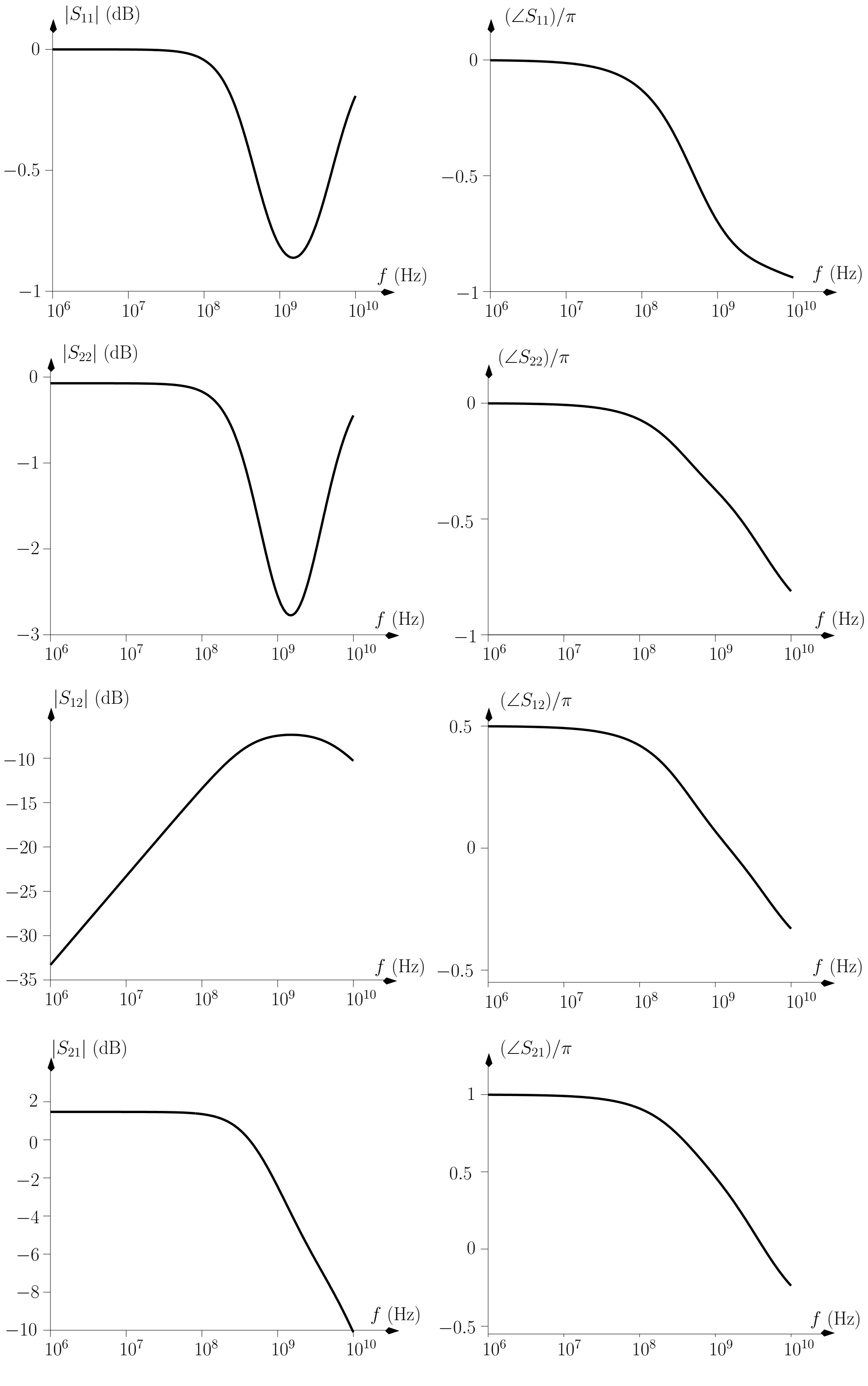


Figure 7.10: *S-parameters (magnitude and phase) plotted versus frequency. a) S_{11}. b) S_{22}. c) S_{12}. d) S_{21}.*

Solution of problem 7.16

The chain parameters can be defined by the set of eqns:

$$\begin{cases} V_1 = A_{11}\, V_2 - A_{12}\, I_2 \\ I_1 = A_{21}\, V_2 - A_{22}\, I_2 \end{cases} \tag{7.21}$$

and the S-parameters can be defined by the set of eqns below:

$$\begin{cases} b_1 = S_{11}\, a_1 + S_{12}\, a_2 \\ b_2 = S_{21}\, a_1 + S_{22}\, a_2 \end{cases} \tag{7.22}$$

that is

$$\begin{cases} (V_1 - Z_o\, I_1) = S_{11}\,(V_1 + Z_o\, I_1) + S_{12}\,(V_2 + Z_o\, I_2) \\ (V_2 - Z_o\, I_2) = S_{21}\,(V_1 + Z_o\, I_1) + S_{22}\,(V_2 + Z_o\, I_2) \end{cases} \tag{7.23}$$

This set of eqns can be rewritten as

$$\begin{cases} V_1\,(1 - S_{11}) = Z_o\,(1 + S_{11})\, I_1 + V_2\, S_{12} + I_2\, Z_o\, S_{12} \\ V_2\,(1 + S_{22}) = V_1\, S_{21} + I_1\, Z_o\, S_{21} + Z_o\,(1 + S_{22})\, I_2 \end{cases} \tag{7.24}$$

Solving the second eqn of 7.24 in order to obtain I_1 we get:

$$I_1 = V_2\,\frac{1 - S_{22}}{Z_o\, S_{21}} - V_1\,\frac{1}{Z_o} - I_2\,\frac{1 + S_{22}}{S_{21}}$$

Substituting I_1 in the first eqn of 7.24 and solving to obtain V_1 we get:

$$V_1 = \frac{(1 - S_{22})(1 + S_{11}) + S_{12}S_{21}}{2S_{21}}\, V_2 - Z_o \frac{(1 + S_{22})(1 + S_{11}) - S_{12}S_{21}}{2S_{21}}\, I_2 \tag{7.25}$$

Substituting V_1, given by the last eqn, in the second eqn of 7.24 and solving in order to obtain I_1 we get:

$$I_1 = \frac{(1 - S_{22})(1 - S_{11}) - S_{12}S_{21}}{2Z_oS_{21}}\, V_2 - \frac{(1 + S_{22})(1 - S_{11}) + S_{12}S_{21}}{2S_{21}}\, I_2 \tag{7.26}$$

Comparing eqns 7.25 and 7.26 with eqn 7.21 we conclude that:

$$\begin{aligned} A_{11} &= \frac{(1 - S_{22})(1 + S_{11}) + S_{12}S_{21}}{2S_{21}} \\ A_{12} &= Z_o \frac{(1 + S_{22})(1 + S_{11}) - S_{12}S_{21}}{2S_{21}} \\ A_{21} &= \frac{(1 - S_{22})(1 - S_{11}) - S_{12}S_{21}}{2Z_oS_{21}} \\ A_{22} &= \frac{(1 + S_{22})(1 - S_{11}) + S_{12}S_{21}}{2S_{21}} \end{aligned}$$

Solution of problem 7.17

We normalise the impedances to $Z_o = 50\ \ \Omega$. Hence we have:

$$
\begin{aligned}
z_1 &= 0.2 - j\,0.6 \\
z_2 &= 1.5 + j\,0.4 \\
z_3 &= 1.2 - j\,0.8 \\
z_4 &= 0.1 - j\,1.4 \\
z_5 &= j\,1 \\
z_6 &= -j\,3.6
\end{aligned}
$$

We represent these normalised impedances on the Smith chart of figure 7.11 and we find the corre-

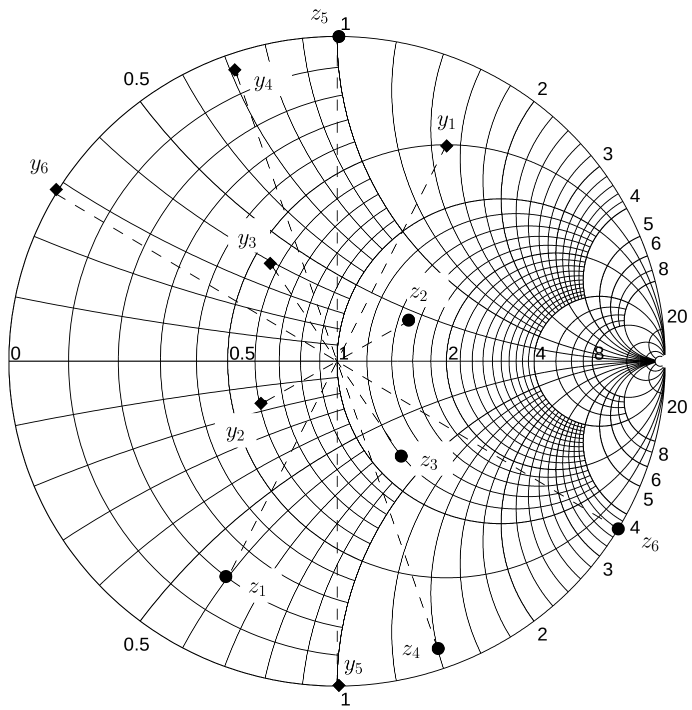


Figure 7.11: *Representation of the normalised impedances.*

sponding normalised admittances:

$$
\begin{aligned}
y_1 &= 0.5 + j\,1.5 \\
y_2 &= 0.62 - j\,0.16 \\
y_3 &= 0.58 + j\,0.38 \\
y_4 &= 0.05 + j\,0.71 \\
y_5 &= -j\,1 \\
y_6 &= j\,0.3
\end{aligned}
$$

The admittances are obtained multiplying each y_k by $Y_o = 1/Z_o$ (20 mS);

$$
\begin{aligned}
Y_1 &= 10 + j\,30\ \ \text{mS} \\
Y_2 &= 12.4 - j\,3.2\ \ \text{mS} \\
Y_3 &= 11.6 + j\,7.6\ \ \text{mS} \\
Y_4 &= 1 + j\,14.2\ \ \text{mS} \\
Y_5 &= -j\,20\ \ \text{mS} \\
Y_6 &= j\,6\ \ \text{mS}
\end{aligned}
$$

Solution of problem 7.18

We normalise the admittances to $Y_o = 1/50 = 20$ mS;

$$\begin{aligned} y_1 &= j\,0.2 \\ y_2 &= 0.08 - j\,3 \\ y_3 &= 0.2 + j\,2 \end{aligned}$$

We represent these normalised admittances on the Smith chart of figure 7.12. From the chart, we

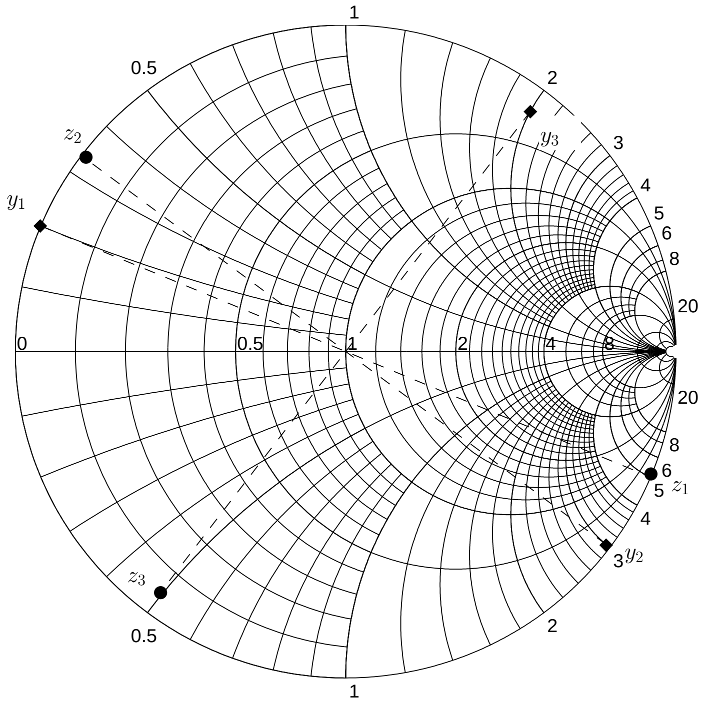


Figure 7.12: *Representation of the normalised admittances.*

find the corresponding normalised impedances:

$$\begin{aligned} z_1 &= -j\,5 \\ z_2 &= 0.009 + j\,0.333 \\ z_3 &= 0.05 - j\,0.5 \end{aligned}$$

The impedances are obtained multiplying each z_k by $Z_o = 1/Y_o = 50$;

$$\begin{aligned} Z_1 &= j\,250\ \Omega \\ Z_2 &= 0.45 + j\,16.7\ \Omega \\ Z_3 &= 2.5 - j\,25\ \Omega \end{aligned}$$

Solution of problem 7.19

Figure 7.13 a) shows the calculation, using the Smith chart, of the L-section which transforms an impedance of $50\ \Omega$ into an impedance $Z_L = 25 - j\,15\ \Omega$. $Z_o = 50\ \Omega$. Figure 7.13 b) shows the L-section obtained.

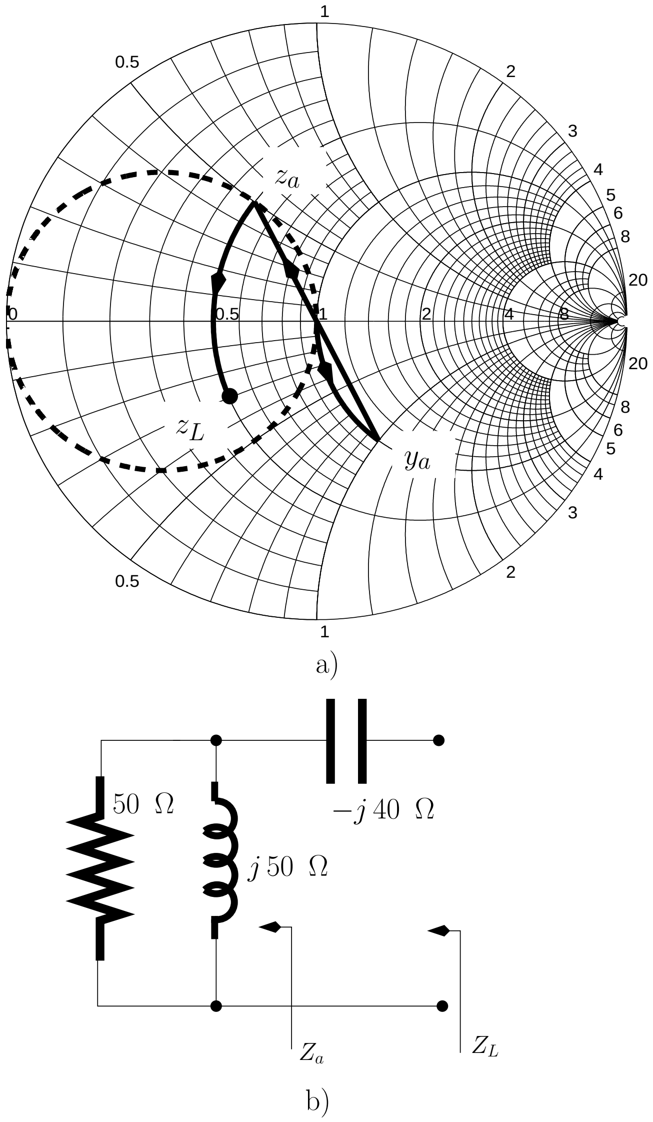


Figure 7.13: *a) Calculation of the L-section using the Smith chart. b) L-section.*

Solution of problem 7.20

Figure 7.14 a) shows S_{11} and S_{22} in the Smith chart (polar coordinates) while figure 7.14 b) shows S_{12} and S_{21} also in polar coordinates. $f_a = (2\,\pi\,R\,C)^{-1}/10$ and $f_b = 10\,(2\,\pi\,R\,C)^{-1}$.

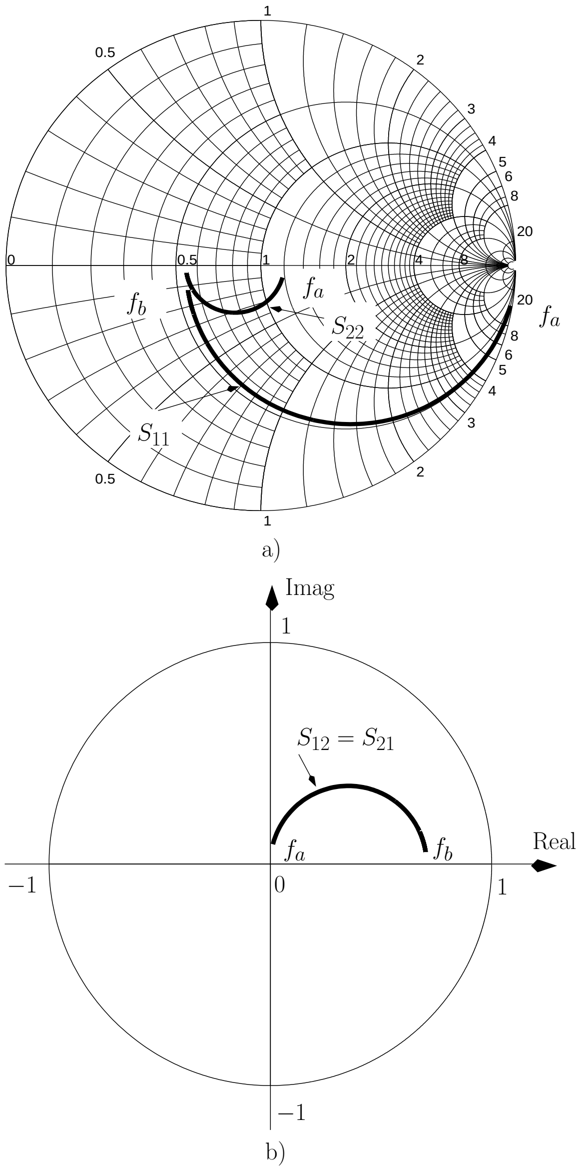
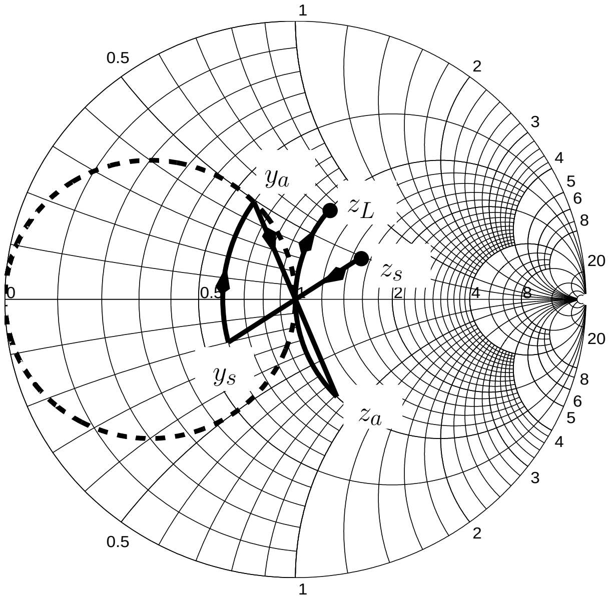


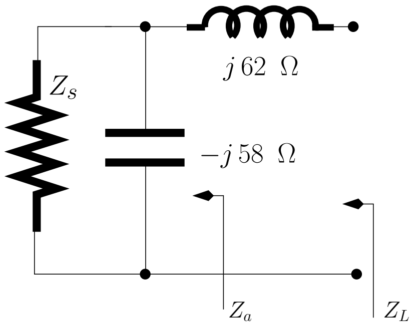
Figure 7.14: *a) S_{11} and S_{22}. b) S_{12} and S_{12}.*

Solution of problem 7.21

Figure 7.15 a) shows the calculation, using the Smith chart, of the L-section which transforms an impedance of $Z_s = 60 + j\,20\,\Omega$ into an impedance $Z_L = 40 + j\,30\ \ \Omega$. $Z_o = 40\,\Omega$. Figure 7.13 b) shows the L-section obtained.



a)



b)

Figure 7.15: *a) Calculation of the L-section using the Smith chart. b) L-section.*

Solution of problem 7.22

Figure 7.16 shows the calculation of the matching circuit using transmission lines. $Z_o = 45\ \Omega$. $l_1 = 0.279\ \lambda$, $l_2 = 0.105\ \lambda$.

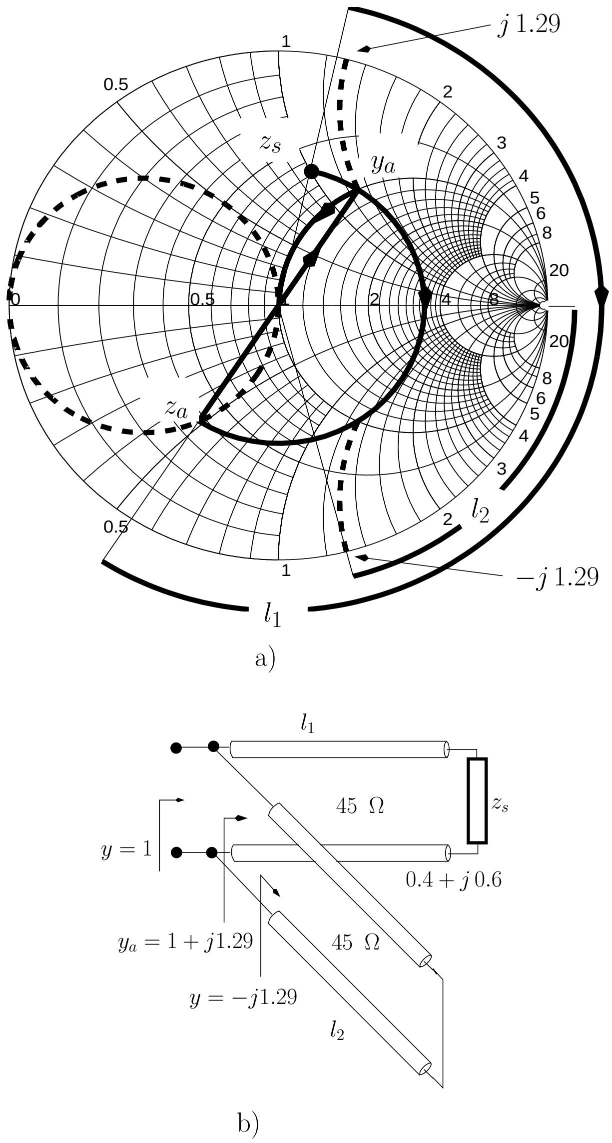


Figure 7.16: *a) Calculation of the matching circuit using the Smith chart. b) The circuit.*

Chapter 8

Noise in electronic circuits

Solution of problem 8.1

- 1. $P[X > 3]$ can be calculated as follows:

$$P[X > 3] = \int_{3}^{\infty} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Using the variable transformation indicated below:

$$\lambda = \frac{x-\mu}{\sigma}$$

we can write $P[X > 3]$ as

$$\begin{aligned} P[X > 3] &= \int_{\frac{3-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{\lambda^2}{2^2}}\, d\lambda \\ &= Q\left(\frac{3-\mu}{\sigma}\right) \\ &= 0.046 \end{aligned}$$

- 2. $P[X > -3]$ can be calculated as follows:

$$\begin{aligned} P[X > -3] &= \int_{-3}^{\infty} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\ &= 1 - \int_{-\infty}^{-3} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \end{aligned}$$

Using the variable transformation $y = -x$ we can write the last eqn as

$$P[X > -3] = 1 - \int_{3}^{\infty} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(y+\mu)^2}{2\sigma^2}}\, dy$$

and now by using the variable transformation

$$\lambda = \frac{y-\mu}{\sigma}$$

we can write $P[X > -3]$ as

$$\begin{aligned} P[X > -3] &= 1 - \int_{\frac{3+\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{\lambda^2}{2^2}}\, d\lambda \\ &= 1 - Q\left(\frac{3+\mu}{\sigma}\right) \\ &= 0.85 \end{aligned}$$

- 3. $P[-2 < X < 3]$ can be calculated as follows:

$$\begin{aligned}
P[-2 < X < 3] &= \int_{-2}^{3} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= 1 - \int_{3}^{\infty} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx - \int_{-\infty}^{-2} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= 1 - Q\left(\frac{3-\mu}{\sigma}\right) - Q\left(\frac{2+\mu}{\sigma}\right) \\
&= 0.68
\end{aligned}$$

- 4. $P[X < -2]$ can be calculated as follows:

$$\begin{aligned}
PX < -2] &= \int_{-\infty}^{-2} \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= Q\left(\frac{2+\mu}{\sigma}\right) \\
&= 0.28
\end{aligned}$$

Solution of problem 8.2

- 1. The mean can be calculated as follows:

$$
\begin{aligned}
\mu &= \frac{1}{5}\int_{-2}^{3} x\,dx \\
&= \frac{1}{5}\left.\frac{x^2}{2}\right|_{-2}^{3} \\
&= \frac{1}{2}
\end{aligned}
$$

- 2. The variance can be calculated as follows:

$$
\begin{aligned}
\sigma^2 &= \frac{1}{5}\int_{-2}^{3}\left(x-\frac{1}{2}\right)^2\,dx \\
&= \frac{1}{5}\left.\frac{\left(x-\frac{1}{2}\right)^3}{3}\right|_{-2}^{3} \\
&= 2.08
\end{aligned}
$$

- The third moment can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}\left[X^3\right] &= \frac{1}{5}\int_{-2}^{3} x^3\,dx \\
&= \frac{1}{5}\left.\frac{x^4}{4}\right|_{-2}^{3} \\
&= 3.25
\end{aligned}
$$

Solution of problem 8.3

The characteristic function can be determined as

$$\begin{aligned} C(2\,\pi\,\alpha) &= \int_{-\infty}^{\infty} e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!}\,\delta(x-k)\,e^{-j\,2\,\pi\,\alpha\,x}\,dx \\ &= e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!}\,e^{-j\,2\,\pi\,\alpha\,k} \end{aligned}$$

The mean can be determined as

$$\begin{aligned} \mathrm{E}\left[X\right] &= \frac{1}{-j\,2\,\pi}\,\frac{d}{d\,\alpha}\,C(2\,\pi\,\alpha)\bigg|_{\alpha=0} \\ &= e^{-\mu} \underbrace{\sum_{k=0}^{\infty} \frac{\mu^k}{k!}\,k}_{\mu\,e^{\mu}} \\ &= \mu = 0.8 \end{aligned}$$

Solution of problem 8.4

The PDF of the random variable resulting from the sum of the three uniform PDFs can be expressed by;

$$P_Y(y) = \begin{cases} \dfrac{(y+3A)^2}{16A^3} & -3A \leq y < -A \\ -\dfrac{(y^2-3A^2)}{8A^3} & -A \leq y \leq A \\ \dfrac{(y-3A)^2}{16A^3} & A < y \leq 3A \\ 0 & \text{elsewhere} \end{cases} \tag{8.1}$$

with $A = 3$. Hence, $P[Y > 4]$ can be written as

$$\begin{aligned} P[Y > 4] &= \int_4^9 \frac{(y-3A)^2}{16A^3} dy \\ &= 96.5 \times 10^{-3} \end{aligned}$$

and $P[Y > 8.9]$ can be written as

$$\begin{aligned} P[Y > 8.9] &= \int_{8.9}^9 \frac{(y-3A)^2}{16A^3} dy \\ &= 77.16 \times 10^{-8} \end{aligned}$$

We use now the central limit theorem to estimate these probabilities. The variance of each uniform random variable is $\sigma^2 = \frac{6^2}{12} = 3$ while the mean is zero. $P[Y > 4]$ can be estimated as follows:

$$\begin{aligned} P[Y > 4] &= Q\left(\frac{4}{\sqrt{3}\sigma}\right) \\ &= 91.2 \times 10^{-3} \end{aligned}$$

and $P[Y > 8.9]$ can be estimated according to

$$\begin{aligned} P[Y > 8.9] &= Q\left(\frac{8.9}{\sqrt{3}\sigma}\right) \\ &= 1.5 \times 10^{-3} \end{aligned}$$

The central limit theorem (CLT) provides a good estimation for $P[Y > 4]$ since the error is about 5.4%. However, for $P[Y > 8.9]$ the CLT provides a poor estimation with a very large error.

Solution of problem 8.5

The mean can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}\left[a(t)\right] &= \int_{-\infty}^{\infty} e^{-3\,x\,t}\,p_X(x)\,dx \\
&= \frac{1}{3}\int_{-1}^{2} e^{-3\,x\,t}\,dx \\
&= \frac{-1}{9\,t}\left(e^{-6\,t} - e^{-3\,t}\right) \\
&= \frac{1}{9\,t}\,e^{-\frac{3\,t}{2}}\,\cosh\left(\frac{9\,t}{2}\right)
\end{aligned}
$$

The autocorrelation function can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}\left[a(t_1)\,a_(t_2\right] &= \int_{-\infty}^{\infty} e^{-3\,x\,(t_1+t_2)}\,p_X(x)\,dx \\
&= \frac{1}{3}\int_{-1}^{2} e^{-3\,x\,(t_1+t_2)}\,dx \\
&= \frac{-1}{9\,(t_1+t_2)}\left(e^{-6\,(t_1+t_2)} - e^{-3\,(t_1+t_2)}\right) \\
&= \frac{1}{9\,(t_1+t_2)}\,e^{-\frac{3\,(t_1+t_2)}{2}}\,\cosh\left(\frac{9\,(t_1+t_2)}{2}\right)
\end{aligned}
$$

Solution of problem 8.6

The power spectral density $S_{vv^*}(f)$ can be obtained calculating the Fourier transform of $R_v(\tau)$, that is

$$S_{vv^*}(f) = \int_{-\infty}^{\infty} R_v(\tau)\, e^{-j\,2\,\pi\,f\,\tau}\, d\,\tau \tag{8.2}$$

$R_v(\tau)$ is a triangular function. Hence, by consulting the Fourier transform table provided in appendix A we obtain:

$$S_{vv^*}(f) = \sigma^2\, A\, \mathrm{sinc}^2\,(f\, A)$$

Solution of problem 8.7

The equivalent noise bandwidth, B_N, can be calculated as

$$B_{N_1} = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega \tag{8.3}$$

- $|H_1(\omega)|^2$ can be written as

$$|H_1(\omega)|^2 = \frac{k_1}{\omega^2 - p_1} + \frac{k_2}{\omega^2 - p_2}$$

with

$$\begin{aligned} p_1 &= (1 - 2\,\eta^2 + 2\,j\,\eta\,\sqrt{1-\eta^2})\,\omega_n^2 \\ &= (\sqrt{1-\eta^2} + j\,\eta)^2\,\omega_n^2 \\ p_2 &= (1 - 2\,\eta^2 - 2\,j\,\eta\,\sqrt{1-\eta^2})\,\omega_n^2 \\ &= (\sqrt{1-\eta^2} - j\,\eta)^2\,\omega_n^2 \\ k_1 &= \frac{-j\,\omega_n^2}{4\,\eta\,\sqrt{1-\eta^2}} \\ k_2 &= \frac{j\,\omega_n^2}{4\,\eta\,\sqrt{1-\eta^2}} \end{aligned}$$

It is known that

$$\int \frac{k_1}{\omega^2 - p_1}\, d\omega = -\frac{K_1}{\sqrt{p_1}} \tanh^{-1}\left(\frac{\omega}{\sqrt{p_1}}\right)$$

It is also known that

$$\lim_{z\to\infty} \tanh^{-1}(z) = \begin{cases} \frac{j\pi}{2} & \text{if Imag}\,[z] > 0 \\ \frac{-j\pi}{2} & \text{if Imag}\,[z] < 0 \end{cases} \tag{8.4}$$

Using the results above we can solve eqn 8.3 to obtain:

$$B_N = \frac{\omega_n}{4\,\eta}$$

- $|H_2(\omega)|^2$ can be written as

$$|H_2(\omega)|^2 = \frac{k_1}{\omega^2 - p_1} + \frac{k_2}{\omega^2 - p_2}$$

with

$$\begin{aligned} p_1 &= (1 - 2\,\eta^2 + 2\,j\,\eta\,\sqrt{1-\eta^2})\,\omega_n^2 \\ &= (\sqrt{1-\eta^2} + j\,\eta)^2\,\omega_n^2 \\ p_2 &= (1 - 2\,\eta^2 - 2\,j\,\eta\,\sqrt{1-\eta^2})\,\omega_n^2 \\ &= (\sqrt{1-\eta^2} - j\,\eta)^2\,\omega_n^2 \\ k_1 &= 4\,\omega_n^2\,\eta^2 \frac{p_1}{p_1 - p_2} \\ k_2 &= 4\,\omega_n^2\,\eta^2 \frac{p_2}{p_2 - p_1} \end{aligned}$$

Using the results above we can solve eqn 8.3 to obtain:

$$B_{N_2} = \omega_n\,\eta$$

- The equivalent noise bandwidth for $H_1(\omega) + H_2(\omega)$ is

$$\begin{aligned} B_{N_3} &= B_{N_1} + B_{N_2} \\ &= \omega_n \left(\eta + \frac{1}{4\,\eta} \right) \end{aligned}$$

Figure 8.1 shows the equivalent noise bandwidths determined above, normalised to ω_n, versus η.

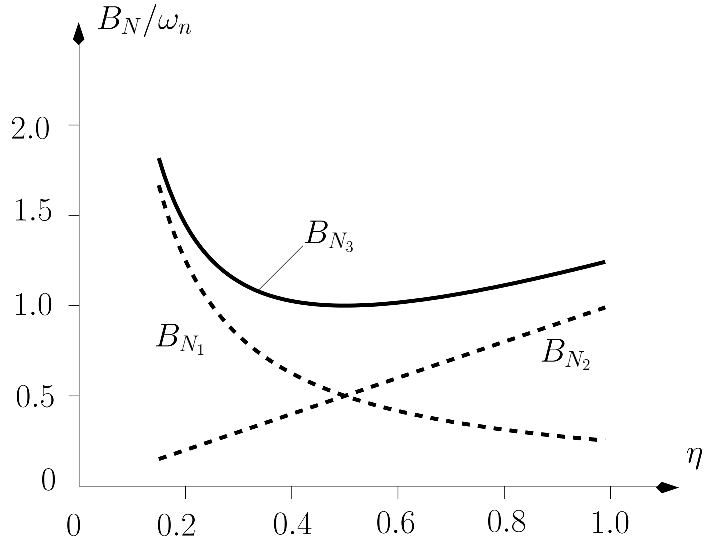


Figure 8.1: *Equivalent noise bandwidths normalised to ω_n versus η.*

Solution of problem 8.8

We assume that the amplifier of figure 8.2 has an input impedance Z_{in} and a current gain $A_i = i_o/i_i$.

- *Contribution of* $\mathbf{u_n}$ *to the output current:* From figure 8.2 b) we can write:

$$i_i = \frac{\mathbf{u_n}}{Z_{in} + Z_s}$$

and

$$\begin{aligned} i_o &= A_i \, i_i \\ &= A_i \, \frac{\mathbf{u_n}}{Z_{in} + Z_s} \end{aligned}$$

- *Contribution of* $\mathbf{i_n}$ *to the output current:* From figure 8.2 c) we can write:

$$i_i = \frac{\mathbf{i_n} \, Z_s}{Z_{in} + Z_s}$$

and

$$\begin{aligned} i_o &= A_i \, i_i \\ &= A_i \, \frac{\mathbf{i_n} \, Z_s}{Z_{in} + Z_s} \end{aligned}$$

- *Contribution of* $\mathbf{i_{ns}}$ *to the output current:* From figure 8.2 d) we can write:

$$i_i = \frac{\mathbf{i_{ns}} \, Z_s}{Z_{in} + Z_s}$$

and

$$\begin{aligned} i_o &= A_i \, i_i \\ &= A_i \, \frac{\mathbf{i_{ns}} \, Z_s}{Z_{in} + Z_s} \end{aligned}$$

Since the total current gain, A_{i_s} is

$$\begin{aligned} A_{i_s} &= \frac{i_o}{i_s} \\ &= A_i \, \frac{Z_s}{Z_{in} + Z_s} \end{aligned}$$

we can obtain the equivalent input noise current as follows:

$$\begin{aligned} \mathbf{i_{n_{eq}}} &= \frac{A_i \, \frac{\mathbf{u_n}}{Z_{in}+Z_s} + A_i \, \frac{\mathbf{i_n} \, Z_s}{Z_{in}+Z_s} + A_i \, \frac{\mathbf{i_{ns}} \, Z_s}{Z_{in}+Z_s}}{A_{i_s}} \\ &= \frac{\mathbf{u_n}}{Z_s} + \mathbf{i_n} + \mathbf{i_{ns}} \end{aligned}$$

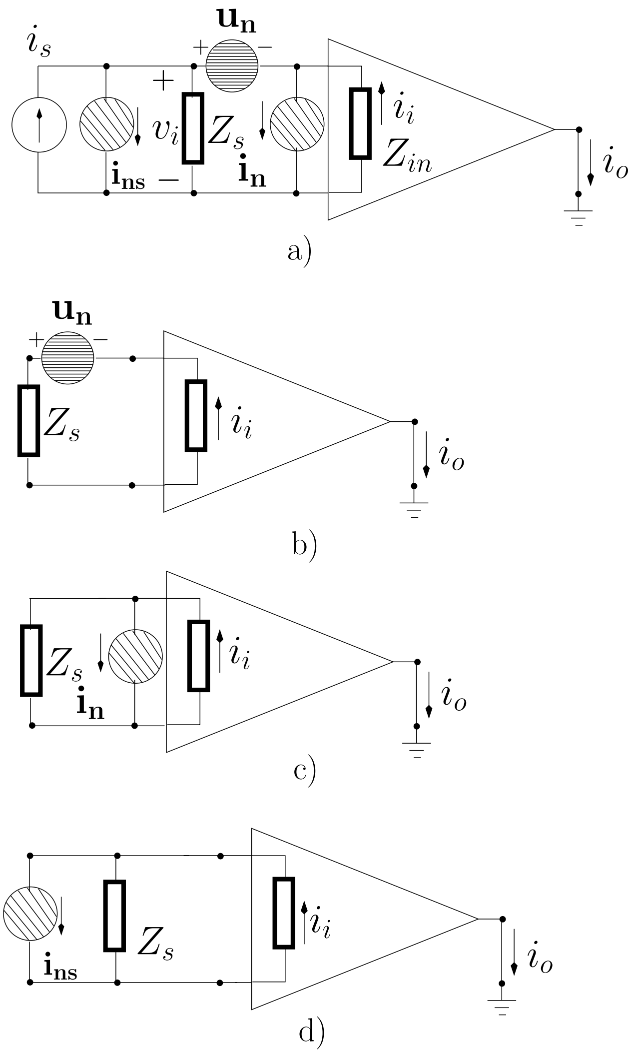


Figure 8.2: *a) Noisy amplifier. a) Equivalent circuit for the calculation of the contribution of* $\mathbf{u_n}$ *to the output current. b) Equivalent circuit for the calculation of the contribution of* $\mathbf{i_n}$ *to the output current. c) Equivalent circuit for the calculation of the contribution of* $\mathbf{i_{ns}}$ *to the output current.*

Solution of problem 8.9

We assume that the amplifier, shown here in figure 8.3, has an input impedance Z_{in} and a voltage gain $A_v = v_o/v_i$.

- *Contribution of $\mathbf{u_n}$ to the output voltage:* From figure 8.3 b) we can write:

$$v_i = \mathbf{u_n}\,\frac{Z_{in}}{Z_{in}+Z_s}$$

and

$$\begin{aligned} v_o &= A_v\,v_i \\ &= A_v\,\mathbf{u_n}\,\frac{Z_{in}}{Z_{in}+Z_s} \end{aligned}$$

- *Contribution of $\mathbf{i_n}$ to the output voltage:* From figure 8.3 c) we can write:

$$v_i = \mathbf{i_n}\,\frac{Z_{in}\,Z_s}{Z_{in}+Z_s}$$

and

$$\begin{aligned} v_o &= A_v\,v_i \\ &= A_v\,\mathbf{i_n}\,\frac{Z_{in}\,Z_s}{Z_{in}+Z_s} \end{aligned}$$

- *Contribution of $\mathbf{u_{ns}}$ to the output voltage:* From figure 8.3 d) we can write:

$$v_i = \mathbf{u_{ns}}\,\frac{Z_{in}}{Z_{in}+Z_s}$$

and

$$\begin{aligned} v_o &= A_v\,v_i \\ &= A_v\,\mathbf{u_{ns}}\,\frac{Z_{in}}{Z_{in}+Z_s} \end{aligned}$$

Since the total voltage gain, A_{v_s} is

$$\begin{aligned} A_{v_s} &= \frac{v_o}{v_s} \\ &= A_v\,\frac{Z_{in}}{Z_{in}+Z_s} \end{aligned}$$

we can obtain the equivalent input noise voltage as follows:

$$\begin{aligned} \mathbf{u_{n_{eq}}} &= \frac{A_v\,\mathbf{u_n}\,\frac{Z_{in}}{Z_{in}+Z_s} + A_v\,\mathbf{i_n}\,\frac{Z_{in}\,Z_s}{Z_{in}+Z_s} + \mathbf{u_{ns}}\,\frac{Z_{in}}{Z_{in}+Z_s}}{A_{v_s}} \\ &= \mathbf{u_n} + \mathbf{i_n}\,Z_s + \mathbf{u_{ns}} \end{aligned}$$

The PSD can now be obtained calculating $\left\langle \mathbf{i_{n_{eq}}}\,\mathbf{i_{n_{eq}}}^* \right\rangle$, that is:

$$\left\langle \mathbf{u_{n_{eq}}}\,\mathbf{u_{n_{eq}}}^* \right\rangle = \left\langle (\mathbf{u_n} + \mathbf{i_n}\,Z_s + \mathbf{u_{ns}})\,(\mathbf{u_n} + \mathbf{i_n}\,Z_s + \mathbf{u_{ns}})^* \right\rangle$$

Since $\mathbf{u_{ns}}$ is not correlated with $\mathbf{u_n}$ and $\mathbf{i_n}$ we can write:

$$\begin{aligned} \langle \mathbf{u_{ns}}\,\mathbf{u_n}^* \rangle &= \langle \mathbf{u_n}\,\mathbf{u_{ns}}^* \rangle = 0 \\ \langle \mathbf{u_{ns}}\,\mathbf{i_n}^* \rangle &= \langle \mathbf{i_n}\,\mathbf{u_{ns}}^* \rangle = 0 \end{aligned}$$

In addition we have:

$$\langle (\mathbf{i_n}\,Z_s)\,\mathbf{u_n}^* \rangle + \langle \mathbf{u_n}\,(\mathbf{i_n}\,Z_s)^* \rangle = 2\,\text{Real}\,\left[\langle \mathbf{u_{ns}}\,\mathbf{i_n}^* \rangle\;Z_s^*\right]$$

Finally, we can write:

$$\left\langle \mathbf{u_{n_{eq}}}\,\mathbf{u_{n_{eq}}}^* \right\rangle = \langle \mathbf{u_n}\mathbf{u_n}^* \rangle + \langle \mathbf{u_{ns}}\mathbf{u_{ns}}^* \rangle + \langle \mathbf{i_n}\mathbf{i_n}^* \rangle\;|Z_s|^2 + 2\,\text{Real}\,\left[\langle \mathbf{u_{ns}}\,\mathbf{i_n}^* \rangle\;Z_s^*\right]$$

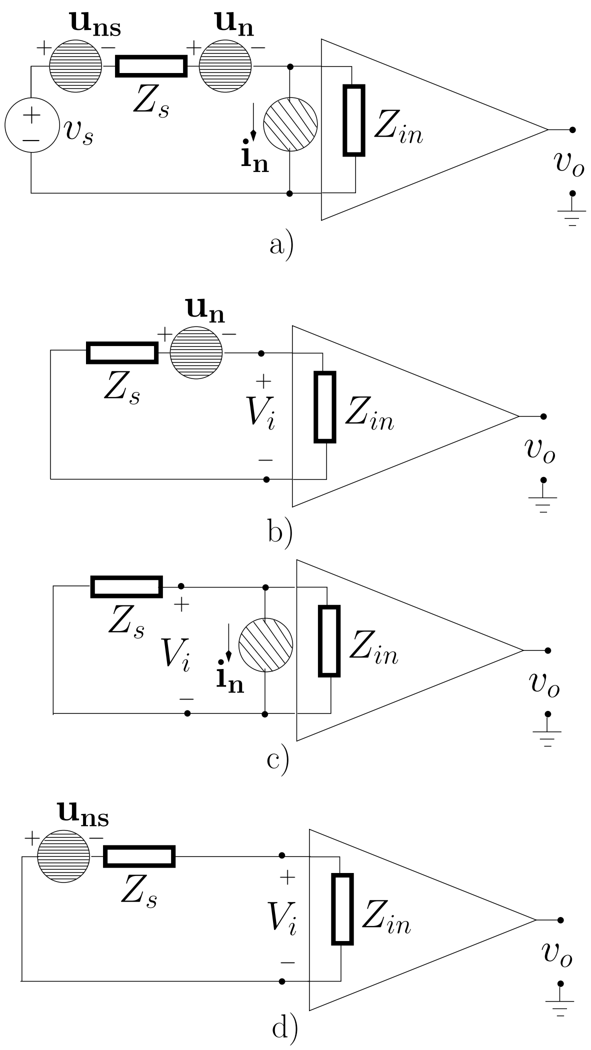


Figure 8.3: *a) Noisy amplifier. a) Equivalent circuit for the calculation of the contribution of* $\mathbf{u_n}$ *to the output voltage. b) Equivalent circuit for the calculation of the contribution of* $\mathbf{i_n}$ *to the output voltage. c) Equivalent circuit for the calculation of the contribution of* $\mathbf{u_{ns}}$ *to the output voltage.*

Solution of problem 8.10

Figure 8.4 a) shows the non-inverting voltage amplifier including the various the noise sources.

- *Contribution of* $\mathbf{u_{R_2}}$ *to the output voltage:* Figure 8.4 b) shows the equivalent circuit. Since there is no voltage difference across R_1 there is no current flowing through R_1. Since the op-amp does not draw any current there is no current flowing through R_2. Hence,we can write

$$v_o = -\mathbf{u_{R_2}} \tag{8.5}$$

- *Contribution of* $\mathbf{u_n}$ *to the output voltage:* From figure 8.4 c) we can write:

$$v_o = \frac{R_2 + R_1}{R_1}\mathbf{u_n} \tag{8.6}$$

- *Contribution of* $\mathbf{i_n}$ *to the output voltage:* From figure 8.4 d) we can write:

$$v_o = -R_2\mathbf{i_n} \tag{8.7}$$

- *Contribution of* $\mathbf{u_{R_1}}$ *to the output voltage:* From figure 8.4 e) we can write:

$$v_o = -\frac{R_2}{R_1}\mathbf{u_{R_1}} \tag{8.8}$$

The equivalent input noise voltage source can be obtained dividing the sum of the contributions calculated above by the voltage gain $1 + R_2/R_1$, that is:

$$\mathbf{u_{eq}} \quad = \quad \frac{-\mathbf{u_{R_2}} + \frac{R_2+R_1}{R_1}\mathbf{u_n} - R_2\mathbf{i_n} - \frac{R_2}{R_1}\mathbf{u_{R_1}}}{1 + \frac{R_2}{R_1}} \tag{8.9}$$

that is

$$\mathbf{u_{eq}} \quad = \quad -\mathbf{u_{R_2}}\frac{R_1}{R_2 + R_1} + \mathbf{u_n} - \frac{R_2\,R_1}{R_2 + R_1}\mathbf{i_n} - \frac{R_1}{R_2 + R_1}\mathbf{u_{R_1}} \tag{8.10}$$

Since all the noise sources are uncorrelated we have:

$$\begin{aligned} \langle \mathbf{u_{eq}}\, \mathbf{u_{eq}}^* \rangle \quad &= \quad \frac{R_1^2}{(R_2 + R_1)^2} \langle \mathbf{u_{R_2}}\, \mathbf{u_{R_2}}^* \rangle + \langle \mathbf{u_n}\, \mathbf{u_n}^* \rangle \\ &+ \quad \frac{(R_2\,R_1)^2}{(R_2 + R_1)^2} \langle \mathbf{i_n}\, \mathbf{i_n}^* \rangle + \frac{R_1^2}{(R_2 + R_1)^2} \langle \mathbf{u_{R_1}}\, \mathbf{u_{R_1}} \rangle \end{aligned} \tag{8.11}$$

For this amplifier we have:

$$\begin{aligned} \langle \mathbf{u_{R_1}}\, \mathbf{u_{R_1}}^* \rangle \quad &= \quad 2\,\mathcal{K}\,\mathcal{T}\,R_1 \\ &= \quad 8.0 \times 10^{-18}\ \ \mathrm{V^2/Hz} \\ \langle \mathbf{u_{R_2}}\, \mathbf{u_{R_2}}^* \rangle \quad &= \quad 2\,\mathcal{K}\,\mathcal{T}\,R_2 \\ &= \quad 7.2 \times 10^{-17}\ \ \mathrm{V^2/Hz} \end{aligned}$$

The PSD of the equivalent input noise voltage is:

$$\langle \mathbf{u_{eq}}\, \mathbf{u_{eq}}^* \rangle \quad = \quad 4.4 \times 10^{-18}\ \ \mathrm{V^2/Hz}$$

Figure 8.5 a) shows the inverting voltage amplifier including the noise sources. The contributions of each noise source to the output voltage are the same as those calculated above, that is:

$$v_o \quad = \quad -\mathbf{u_{R_2}} + \frac{R_2 + R_1}{R_1}\mathbf{u_n} - R_2\mathbf{i_n} - \frac{R_2}{R_1}\mathbf{u_{R_1}} \tag{8.12}$$

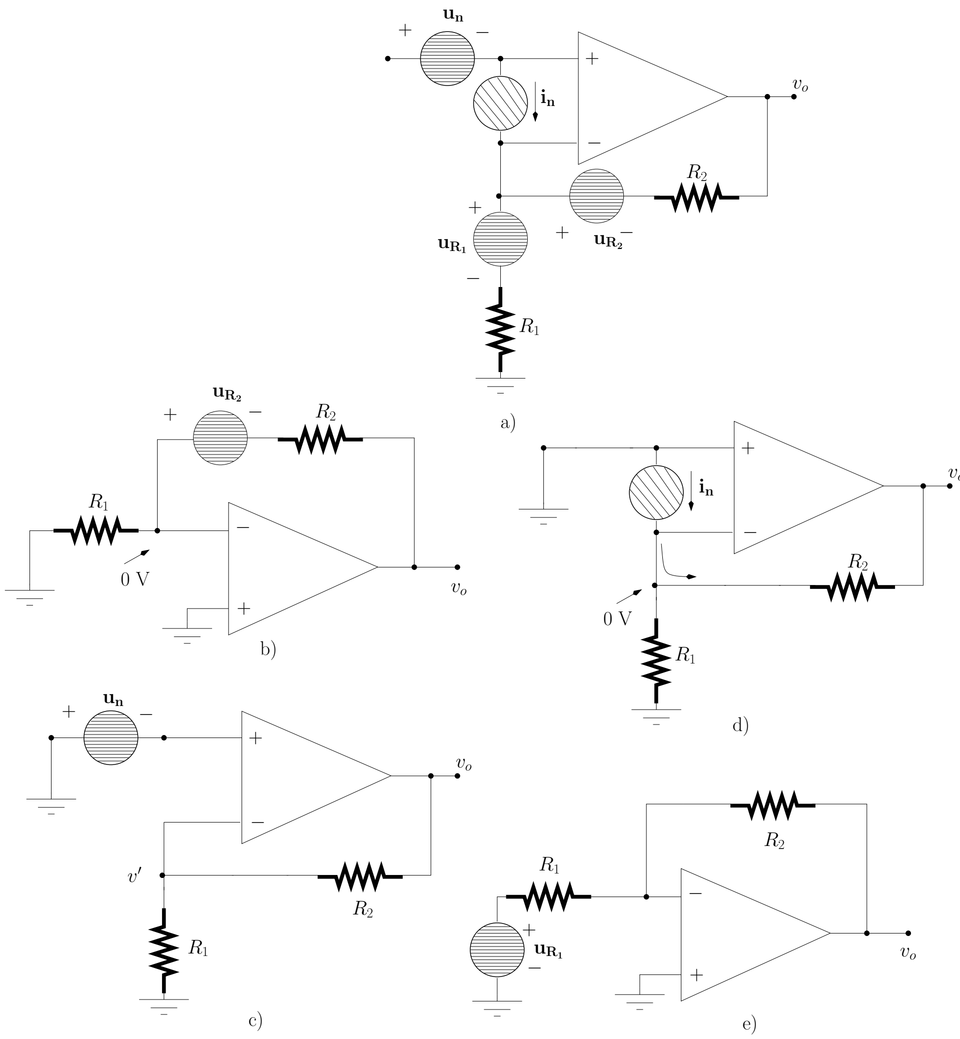


Figure 8.4: *a) Non-inverting amplifier. b) Contribution of* $\mathbf{u_{R_2}}$ *to the output voltage. c) Contribution of* $\mathbf{u_n}$ *to the output voltage. d) Contribution of* $\mathbf{i_n}$ *to the output voltage. e) Contribution of* $\mathbf{u_{R_1}}$ *to the output voltage.*

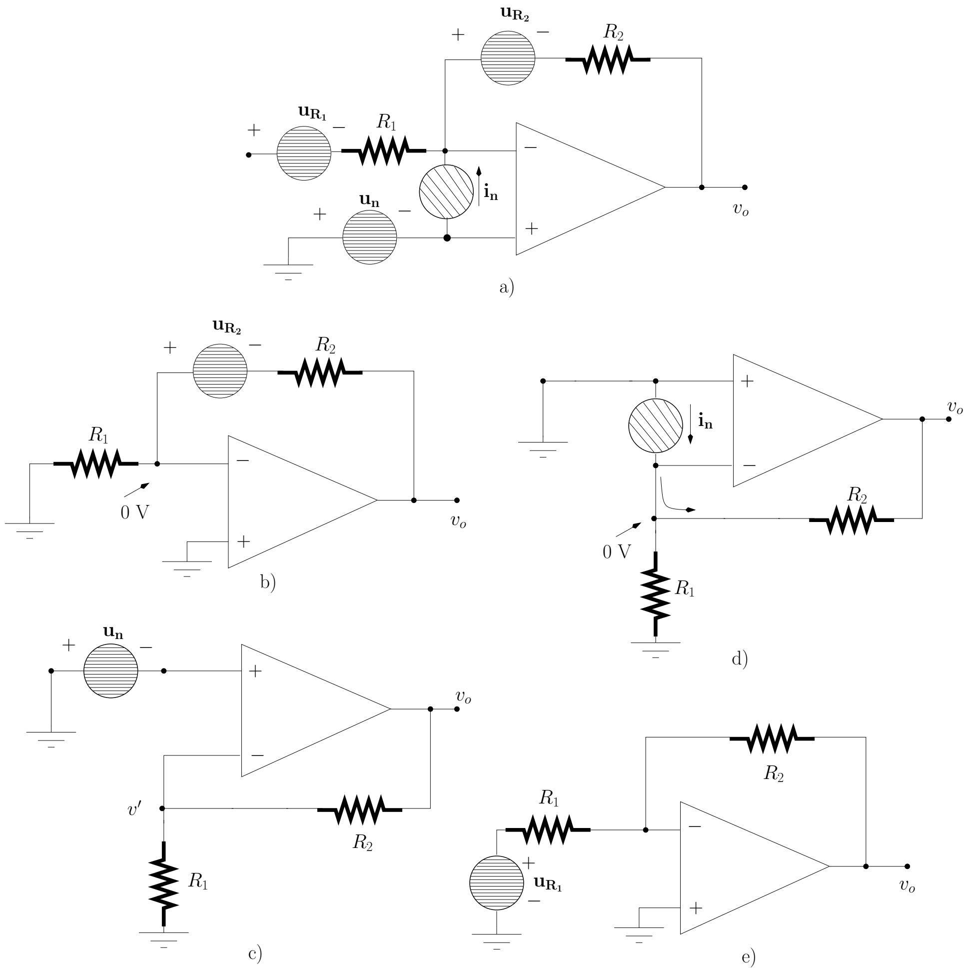


Figure 8.5: *a) Inverting amplifier. b) Contribution of $\mathbf{u_{R_2}}$ to the output voltage. c) Contribution of $\mathbf{u_n}$ to the output voltage. d) Contribution of $\mathbf{i_n}$ to the output voltage. e) Contribution of $\mathbf{u_{R_1}}$ to the output voltage.*

The equivalent input noise voltage source can be obtained dividing the sum of the contributions calculated above by the voltage gain $-R_2/R_1$, that is:

$$\mathbf{u_{eq}} = \frac{-\mathbf{u_{R_2}} + \frac{R_2+R_1}{R_1}\mathbf{u_n} - R_2\mathbf{i_n} - \frac{R_2}{R_1}\mathbf{u_{R_1}}}{-\frac{R_2}{R_1}} \tag{8.13}$$

that is

$$\mathbf{u_{eq}} = \mathbf{u_{R_2}}\frac{R_1}{R_2} - \mathbf{u_n}\frac{R_2+R_1}{R_2} + R_1\mathbf{i_n} + \mathbf{u_{R_1}} \tag{8.14}$$

Since all the noise sources are uncorrelated we have:

$$\begin{aligned}
\langle \mathbf{u_{eq}}\,\mathbf{u_{eq}}^* \rangle &= \langle \mathbf{u_{R_2}}\,\mathbf{u_{R_2}}^* \rangle \frac{R_1^2}{R_2^2} + \langle \mathbf{u_n}\mathbf{u_n}^* \rangle \frac{(R_2+R_1)^2}{R_2^2} \\
&+ \langle \mathbf{i_n}\mathbf{i_n}^* \rangle R_1^2 + \langle \mathbf{u_{R_1}}\,\mathbf{u_{R_1}}^* \rangle \\
&= 1.3 \times 10^{-17}\ \text{V}^2/\text{Hz}
\end{aligned}$$

Solution of problem 8.11

The noise factor can be expressed as

$$F \quad = \quad 1 + \frac{\langle \mathbf{i_n}\ \mathbf{i_n^*} \rangle + 2\,\text{Real}[\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle\ Y_s(\omega)] + \langle \mathbf{u_n}\ \mathbf{u_n^*} \rangle\, |Y_s(\omega)|^2}{2\,\mathcal{K}\,\mathcal{T}\ \text{Real}[Y_s(\omega)]}$$

that is;

$$F \quad = \quad 1 + \frac{g_n}{G_s} + \frac{R_n\,(G_s^2 + W_s^2)}{G_s} + 2\,\frac{\text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]}{2\,\mathcal{K}\,\mathcal{T}} - 2\,\frac{\text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]\ W_s}{2\,\mathcal{K}\,\mathcal{T}\,G_s} \tag{8.15}$$

with:

$$\begin{aligned} 2\,\mathcal{K}\,\mathcal{T}\,g_n &= \langle \mathbf{i_n}\ \mathbf{i_n^*} \rangle \\ 2\,\mathcal{K}\,\mathcal{T}\,R_n &= \langle \mathbf{u_n}\ \mathbf{u_n^*} \rangle \\ Y_s(\omega) &= G_s + j\,W_s \end{aligned}$$

The optimum value for W_s can be obtained by differentiating eqn 8.15 and setting the differential to zero, that is

$$\begin{aligned} \frac{d\,F}{d\,W_s} &= 0 \\ \Leftrightarrow \quad W_{s_{opt}} &= \frac{\text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]}{2\,\mathcal{K}\,\mathcal{T}\,R_n} \end{aligned} \tag{8.16}$$

The optimum value for G_s can be obtained by differentiating eqn 8.15 and setting the differential to zero, that is

$$\begin{aligned} \frac{d\,F}{d\,G_s} &= 0 \\ \Leftrightarrow \quad G_{s_{opt}} &= \sqrt{\frac{g_n + R_n\,W_s^2 - 2\,\frac{\text{Imag}\,[\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]}{2\,\mathcal{K}\,\mathcal{T}}\,W_s}{R_n}} \end{aligned} \tag{8.17}$$

Using the result in eqn 8.16 we obtain:

$$\begin{aligned} G_{s_{opt}} &= \sqrt{\frac{g_n - R_n\,W_{s_{opt}}^2}{R_n}} \\ \Leftrightarrow \quad g_n &= R_n\,(G_{s_{opt}}^2 + W_{s_{opt}}^2) \\ \Leftrightarrow \quad g_n &= R_n\,|Y_{s_{opt}}|^2 \end{aligned}$$

Solution of problem 8.12

It is known that

$$\begin{aligned} \langle \mathbf{i_n}\ \mathbf{i_n^*} \rangle &= 2\,\mathcal{K}\,\mathcal{T}\,g_n \\ \langle \mathbf{u_n}\ \mathbf{u_n^*} \rangle &= 2\,\mathcal{K}\,\mathcal{T}\,R_n \end{aligned}$$

and $\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle = \langle \mathbf{i_n}\ \mathbf{i_n^*} \rangle^*$. The quantity $\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle$ satisfies the following eqn:

$$\begin{aligned} F_{min} &= 1 + \frac{2\,\mathcal{K}\,\mathcal{T}\,g_n}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} + \frac{2\,\mathcal{K}\,\mathcal{T}\,R_n\,|Y_{s_{opt}}|^2}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} \\ &+ \frac{2\,G_{s_{opt}}\ \text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] - 2\,W_{s_{opt}}\ \text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} \\ &= 1 + \frac{4\,\mathcal{K}\,\mathcal{T}\,g_n}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} + \frac{2\,G_{s_{opt}}\ \text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] - 2\,W_{s_{opt}}\ \text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} \end{aligned} \tag{8.18}$$

From problem 8.11 it is known that:

$$\text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] = 2\,\mathcal{K}\,\mathcal{T}\,R_n\,W_{s_{opt}} \tag{8.19}$$

Now eqn. 8.18 can be written as follows:

$$F_{min} = 1 + \frac{4\,\mathcal{K}\,\mathcal{T}\,g_n}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} + \frac{2\,G_{s_{opt}}\ \text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] - 4\,\mathcal{K}\,\mathcal{T}\ W_{s_{opt}}^2\ R_n}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} \tag{8.20}$$

It is also known from problem 8.11, that

$$W_{s_{opt}}^2\,R_n = g_n - R_n\,G_{s_{opt}}^2$$

Using this result in eqn 8.20 we can write:

$$F_{min} = 1 + \frac{4\,\mathcal{K}\,\mathcal{T}\,g_n}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}} + \frac{2\,G_{s_{opt}}\ \text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] - 4\,\mathcal{K}\,\mathcal{T}\ (g_n - R_n\,G_{s_{opt}}^2)}{2\,\mathcal{K}\,\mathcal{T}\,G_{s_{opt}}}$$

Solving this eqn to obtain Real $[\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle]$ we get;

$$\text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] = 2\,\mathcal{K}\,\mathcal{T}\ \left[\frac{F_{min} - 1}{2} - R_n\,G_{s_{opt}}\right] \tag{8.21}$$

Using the result of eqn 8.19 we can now write

$$\text{Real}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] + j\ \text{Imag}\ [\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle] = 2\,\mathcal{K}\,\mathcal{T}\ \left[\frac{F_{min} - 1}{2} - R_n\,(G_{s_{opt}} - j\,W_{s_{opt}})\right]$$

that is:

$$\langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle = 2\,\mathcal{K}\,\mathcal{T}\ \left[\frac{F_{min} - 1}{2} - R_n\,Y_{s_{opt}}^*\right]$$

and

$$\begin{aligned} \langle \mathbf{u_n}\ \mathbf{i_n^*} \rangle &= \langle \mathbf{i_n}\ \mathbf{u_n^*} \rangle^* \\ &= 2\,\mathcal{K}\,\mathcal{T}\ \left[\frac{F_{min} - 1}{2} - R_n\,Y_{s_{opt}}\right] \end{aligned}$$

Solution of problem 8.13

- *DC analysis:* For this circuit we can write:

$$\begin{cases} I_D = \frac{1}{2} K_n \frac{W}{L} (V_{GS} - V_{Th})^2 \\ V_D = V_{GS} \\ V_D = V_{CC} - R_D I_D \end{cases} \tag{8.22}$$

Solving we obtain

$$I_D = 1.1 \;\text{ mA} \quad \text{and} \quad V_{GS} = 4.4 \;\text{ V}$$

- *AC and noise analysis:*

$$\begin{aligned} g_m &= \frac{2 I_D}{V_{GS} - V_{Th}} \\ &= 0.76 \text{ mS} \\ r_o &= \frac{V_A}{I_D} \\ &= 45.4 \;\Omega \end{aligned}$$

Figure 8.6 shows the AC equivalent circuit including the various noise sources. The FET intrinsic noise sources are characterised by PSDs given by:

$$\begin{aligned} \langle \mathbf{i_{nd}} \; \mathbf{i_{nd}}^* \rangle &= 2 \mathcal{K} \mathcal{T} \frac{2}{3} g_m \\ \langle \mathbf{i_{nf}} \; \mathbf{i_{nf}}^* \rangle &= 2 \mathcal{K} \mathcal{T} \frac{2}{3} g_m \frac{f_c}{f} \\ \langle \mathbf{i_{ng}} \; \mathbf{i_{ng}}^* \rangle &= q \, I_G \end{aligned}$$

These sources are uncorrelated.

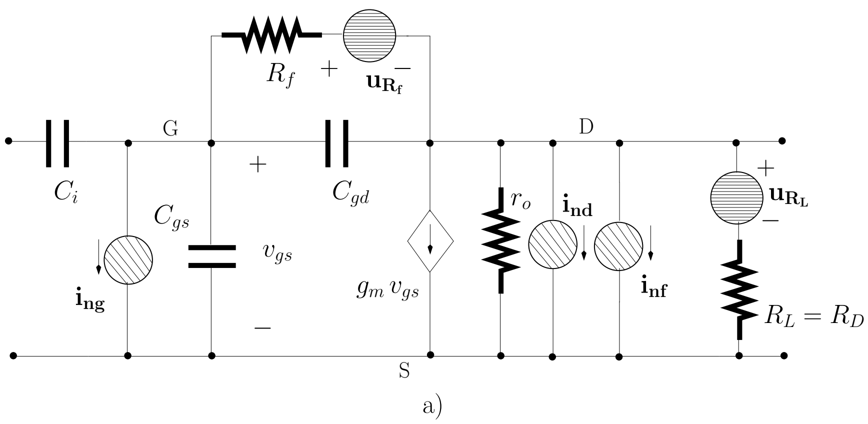


Figure 8.6: *Amplifier AC equivalent circuit*

The load impedance, R_L, represented in figure 8.6 is effectively equal to the drain impedance; $R_L = R_D = 10$ kΩ. The noise voltage sources of R_f and R_L are characterised by PSDs given by:

$$\begin{aligned} \langle \mathbf{i_{R_L}} \; \mathbf{i_{R_L}}^* \rangle &= 2 \mathcal{K} \mathcal{T} R_L \\ \langle \mathbf{u_{R_f}} \; \mathbf{i_{R_f}}^* \rangle &= 2 \mathcal{K} \mathcal{T} R_f \end{aligned}$$

Figure 8.7 shows the amplifier as the interconnection of elementary two-ports circuits. $\mathbf{Y_{gd}}$ can be

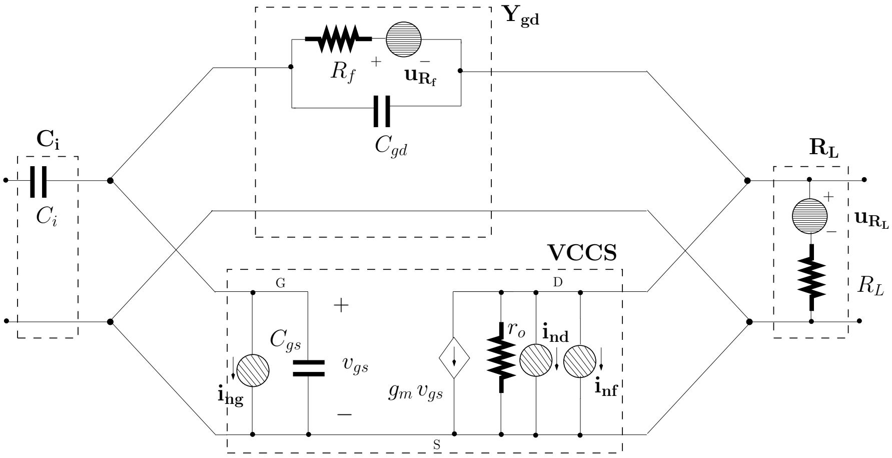


Figure 8.7: *The amplifier as an interconnection of two-ports*

characterised by an admittance representation as follows (see also appendix C):

$$\left[\boldsymbol{Y}_{\mathbf{Y_{gd}}}\right] = \begin{bmatrix} Y_{gd} & -Y_{gd} \\ -Y_{gd} & Y_{gd} \end{bmatrix} \tag{8.23}$$

where Y_{gd} corresponds to the admittance of the parallel connection of C_{gd} with R_f:

$$Y_{gd} = \frac{1}{R_f} + j\,\omega\,C_{gd} \tag{8.24}$$

$$\left[\mathbf{C}_{\mathbf{Y_{Y_{gd}}}}\right] = 2\,\mathcal{K}\,\mathcal{T} \begin{bmatrix} R_f^{-1} & -R_f^{-1} \\ -R_f^{-1} & R_f^{-1} \end{bmatrix} \tag{8.25}$$

The admittance representation for **VCCS** is given by:

$$[\boldsymbol{Y}_{\mathbf{VCCS}}] = \begin{bmatrix} j\,\omega\,C_{gs} & 0 \\ g_m & r_o^{-1} \end{bmatrix} \tag{8.26}$$

and

$$[\mathbf{C}_{\mathbf{Y_{VCCS}}}] = \begin{bmatrix} q\,I_G & 0 \\ 0 & 2\,\mathcal{K}\,\mathcal{T}\,\frac{2}{3}\,g_m\left(1+\frac{\omega}{\omega_c}\right) \end{bmatrix} \tag{8.27}$$

where $\omega_c = 2\,\pi\,f_c$.

The two-port circuit describing the parallel connection of **VCCS** with $\mathbf{Y_{gd}}$ is designated by **FET** and can be characterised by an admittance representation given by:

$$[\boldsymbol{Y}_{\mathbf{FET}}] = \left[\boldsymbol{Y}_{\mathbf{C_{gd}}}\right] + [\boldsymbol{Y}_{\mathbf{VCCS}}] \tag{8.28}$$

$$[\mathbf{C}_{\mathbf{Y_{FET}}}] = [\mathbf{C}_{\mathbf{Y_{Y_{gd}}}}] + [\mathbf{C}_{\mathbf{Y_{VCCS}}}] \tag{8.29}$$

that is

$$[\boldsymbol{Y}_{\mathbf{FET}}] = \begin{bmatrix} j\,\omega\,C_{gs} + Y_{gd} & -Y_{gd} \\ g_m - Y_{gd} & r_o^{-1} + Y_{gd} \end{bmatrix} \tag{8.30}$$

$$[\mathbf{C}_{\mathbf{Y_{FET}}}] = \begin{bmatrix} q\,I_G + 2\,\mathcal{K}\,\mathcal{T}\,R_f^{-1} & -2\,\mathcal{K}\mathcal{T}\,R_f^{-1} \\ -2\,\mathcal{K}\,\mathcal{T}\,R_f^{-1} & 2\,\mathcal{K}\,\mathcal{T}\left[\frac{2}{3}\,g_m\left(1+\frac{\omega_c}{\omega}\right) + R_f^{-1}\right] \end{bmatrix} \tag{8.31}$$

The two-port $\mathbf{C_i}$ is connected in chain with $\mathbf{FET}$ and with $\mathbf{R_L}$. $\mathbf{C_i}$ can be described by a chain representation as follows:

$$[\boldsymbol{A}_{\mathbf{C_i}}] = \begin{bmatrix} 1 & (j\,\omega\,C_i)^{-1} \\ 0 & 1 \end{bmatrix} \tag{8.32}$$

We consider this capacitor as an ideal element such that $\left[\mathbf{C}_{\mathbf{A}_{\mathbf{C_i}}}\right] = [\mathbf{0}]$ with $[\mathbf{0}]$ representing the null matrix.

The chain representation for $\mathbf{R_L}$ is given by :

$$[\boldsymbol{A}_{\mathbf{R_L}}] = \begin{bmatrix} 1 & 0 \\ R_L^{-1} & 1 \end{bmatrix} \tag{8.33}$$

$$\left[\mathbf{C}_{\mathbf{A}_{\mathbf{R_L}}}\right] = 2\,\mathcal{K}\,\mathcal{T} \begin{bmatrix} 0 & 0 \\ 0 & R_L^{-1} \end{bmatrix} \tag{8.34}$$

It is convenient to represent the two-port $\mathbf{FET}$ using a chain representation which can be obtained as follows:

$$[\boldsymbol{A}_{\mathbf{FET}}] = \begin{bmatrix} \frac{r_o^{-1}+Y_{gd}}{Y_{gd}-g_m} & \frac{1}{Y_{gd}-g_m} \\ \frac{(j\,\omega\,C_{gs}+Y_{gd})(r_o^{-1}+Y_{gd})}{Y_{gd}-g_m} - Y_{gd} & \frac{j\,\omega\,C_{gs}+Y_{gd}}{Y_{gd}-g_m} \end{bmatrix} \tag{8.35}$$

and

$$[\mathbf{C}_{\mathbf{A}_{\mathbf{FET}}}] \;=\; [\mathbf{T}_{(\mathbf{Y}\to\mathbf{A})_{\mathbf{FET}}}]\;[\mathbf{C}_{\mathbf{Y}_{\mathbf{FET}}}]\;[\mathbf{T}_{(\mathbf{Y}\to\mathbf{A})_{\mathbf{FET}}}]^{+} \tag{8.36}$$

with

$$[\mathbf{T}_{(\mathbf{Y}\to\mathbf{A})_{\mathbf{FET}}}] = \begin{bmatrix} 0 & \frac{1}{Y_{gd}-g_m} \\ 1 & \frac{j\,\omega\,C_{gs}+Y_{gd}}{Y_{gd}-g_m} \end{bmatrix} \tag{8.37}$$

The amplifier can now be characterised according to a chain representation as follows:

$$\begin{aligned} [\boldsymbol{A}_{\mathbf{AMP}}] &= [\boldsymbol{A}_{\mathbf{C_i}}] \times [\boldsymbol{A}_{\mathbf{FET}}] \times [\boldsymbol{A}_{\mathbf{R_L}}] \\ [\mathbf{C}_{\mathbf{A_{aux}}}] &= [\boldsymbol{A}_{\mathbf{FET}}] \times \left[\mathbf{C}_{\mathbf{A}_{\mathbf{R_L}}}\right] \times [\boldsymbol{A}_{\mathbf{FET}}]^{+} + [\mathbf{C}_{\mathbf{A}_{\mathbf{FET}}}] \\ [\mathbf{C}_{\mathbf{A}_{\mathbf{AMP}}}] &= [\boldsymbol{A}_{\mathbf{C_i}}] \times [\mathbf{C}_{\mathbf{A_{aux}}}] \times [\boldsymbol{A}_{\mathbf{C_i}}]^{+} \end{aligned} \tag{8.38}$$

The equivalent input noise sources PSDs can now be obtained as follows:

$$\begin{aligned} \langle \mathbf{u_{eq}}\ \mathbf{u_{eq}}^* \rangle &= C_{A_{AMP_{11}}} \\ \langle \mathbf{i_{eq}}\ \mathbf{i_{eq}}^* \rangle &= C_{A_{AMP_{22}}} \\ \langle \mathbf{u_{eq}}\ \mathbf{i_{eq}}^* \rangle &= C_{A_{AMP_{12}}} \end{aligned}$$

Figure 8.8 shows the rms values of these sources versus the frequency. The noise factor can be calculated as indicated below:

$$F \;=\; 1 + \frac{[\boldsymbol{Y_s}]\,[\mathbf{C}_{\mathbf{A}_{\mathbf{AMP}}}]\,[\boldsymbol{Y_s}]^*}{2\,\mathcal{K}\,\mathcal{T}\ \mathrm{Real}\,[Y_s]}$$

where Y_s is the source output admittance and is equal to $(2\ \mathrm{k}\Omega)^{-1} = 0.5$ mS.

$$[\boldsymbol{Y_s}] \;=\; [Y_s\ \ 1]$$

and

$$[\boldsymbol{Y_s}]^* = \begin{bmatrix} Y_s^* \\ 1 \end{bmatrix} \tag{8.39}$$

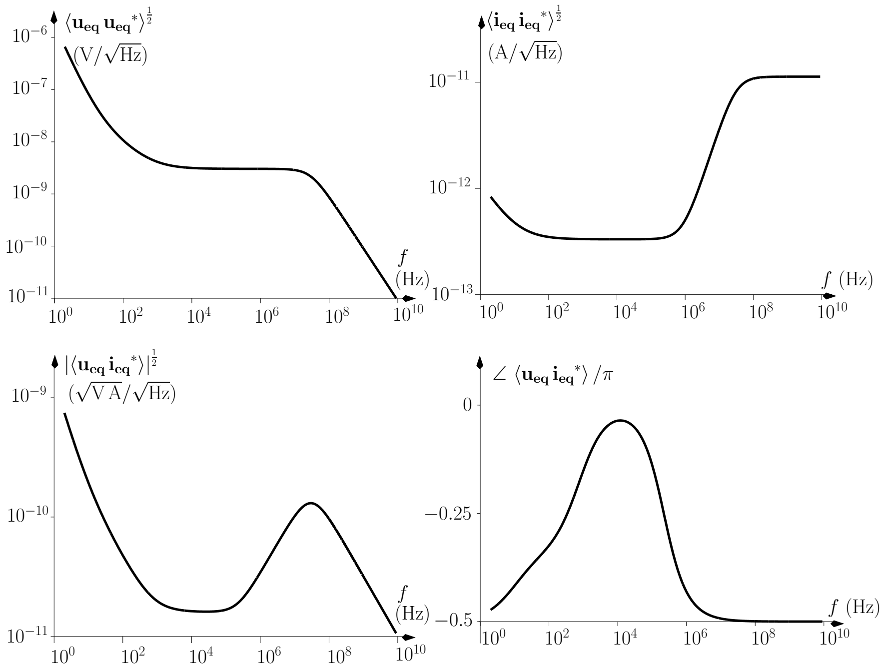


Figure 8.8: *a)* $\langle \mathbf{u_{eq}}\ \mathbf{u_{eq}}^* \rangle^{\frac{1}{2}}$. *b)* $\langle \mathbf{i_{eq}}\ \mathbf{i_{eq}}^* \rangle^{\frac{1}{2}}$. *c)* $\langle \mathbf{u_{eq}}\ \mathbf{i_{eq}}^* \rangle^{\frac{1}{2}}$. *d)* $\angle \langle \mathbf{i_{eq}}\ \mathbf{i_{eq}}^* \rangle$.

Figure 8.9 shows the noise figure (in dB) versus the frequency. It is interesting to note that the noise figure is a strong function of frequency and features minimum values in the frequency range 1 kHz to 2 MHz. This range of frequencies would be the operating range recommended for the amplifier.

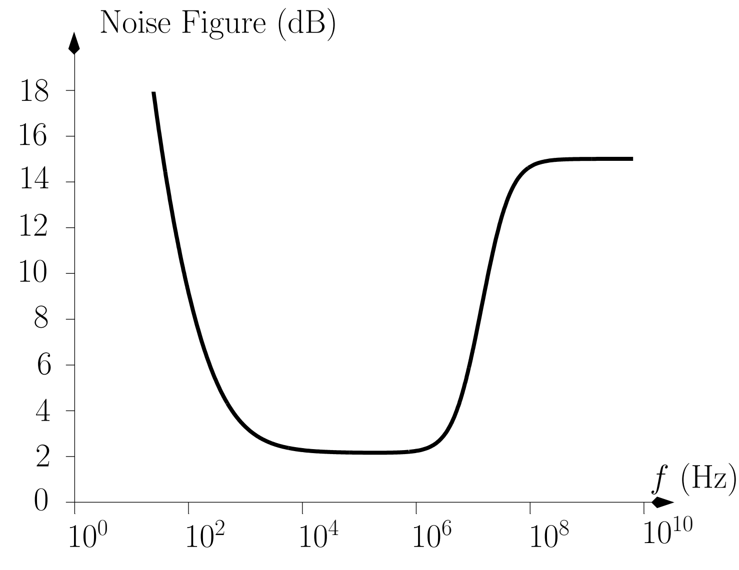


Figure 8.9: *Noise figure.*